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Compact Metrizable Structures via Projective Fraïssé Theory

With an Application to the Study of Fences

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Introduction

In this dissertation we explore projective Fraïssé theory and its applications, as well as limitations, to the study of compact metrizable spaces. The goal of projective Fraïssé theory is to approximate spaces via classes of finite structures and glean topological or dynamical properties of the spaces by relating them to combinatorial features of the class of structures. Using the framework of compact metrizable structures, we establish general results which expand and help contextualize previous works in the field, and apply them to study a class of one-dimensional spaces, which we call fences. We isolate a class of finite structures whose projective Fraïssé limit approximates a distinctive fence — the Fraïssé fence — which we characterize topologically. We explore homogeneity and universality features of the Fraïssé fence and the properties of its endpoints, and provide some results on the dynamics of its group of homeomorphisms.

Projective Fraïssé theory was developed in the wake of the seminal paper [KPT05] by Kechris, Pestov, and Todorcevic, which established a link between topological dynamics, Fraïssé theory, and Ramsey theory. If G is a topological group, a *G-flow* is a compact space X together with a continuous G-action. A G-flow is *minimal* if every orbit is dense. A key result of abstract topological dynamics is that each topological group G admits a *universal minimal flow* — or UMF for short — $\mathrm{M}(G)$, which is, furthermore, unique [Ell60]. The UMF of a topological group G is an interesting topological invariant of G, and its study has attracted widespread attention from a diverse array of fields.

In various circumstances, $\mathrm{M}(G)$ is known to be non-metrizable. This is the case for countable discrete groups and locally compact non-compact groups. On the other hand, there are non-trivial groups whose UMF is a singleton. These groups are called *extremely amenable*, since it follows that their flows admit a fixed point. A link between extreme amenability and Ramsey-type phenomena was noticed in [Pes02], and a general framework for the case of automorphism groups of countable structures was established in [KPT05] (subsequently generalized by Nguyen Van Thé in [NVT13]). The authors use Ramsey theoretical notions to characterize extremely amenable automorphism groups of (direct) Fraïssé structures and give sufficient conditions — later proved to be necessary by Zucker in [Zuc16] — for metrizability of the UMF, of which they also provide an explicit description.

A long-standing open question is whether $\mathrm{M}(\mathrm{Homeo}(P)) \cong P$, holds for the *pseudo-arc* P, a homogeneous one-dimensional continuum, [Usp00]. In [IS06], Irwin and Solecki presented a dualization of usual Fraïssé theory which they employed to study the pseudo-arc, obtaining, among other results, a novel characterization of the space. Projective Fraïssé theory, as it was dubbed, has since been extensively employed to investigate topological and dynamical properties of compact metrizable spaces, see [Cam10, Kwi12, Kwi14, PS18, PS20] for examples beyond the ones explored below.

Typically, the compact metrizable spaces under consideration are realized as a quotient of a projective limit of finite structures in a relational language which contains a binary relation symbol R, whose interpretation on the limit is the equivalence relation which gives rise to the quotient. The limit is called the *prespace*, and can be understood as the combinatorial model of its quotient. On the finite structures, R is not forced to be an equivalence relation. Indeed, in most cases it is symmetric and reflexive but not transitive: the structures can be seen as reflexive graphs with additional structure.

We use the notion of *compact metrizable structure*, introduced in [RZ18], to give a unified presentation of prespaces and their quotients. These are compact metrizable spaces which are also $\mathcal{L}$-structures, in a relational language $\mathcal{L}$, such that the interpretations of the relation symbols are closed sets. The maps between compact metrizable structures in which we are interested are *epimorphisms*, continuous surjections such that the structure of the codomain is the image of the structure of the domain.

In Chapter 1 we present the theory, we introduce novel notions and establish combinatorial criteria which are of general interest, and we test the scope and limits of this approach. Lemma 1.2.3 characterizes which projective sequences of structures in a language containing a binary relation symbol R have limits on which R is an equivalence relation, and Lemma 1.3.4 gives conditions under which the resulting quotient map is irreducible. The irreducibility condition entails a correspondence between structures in the projective sequence and regular quasi-partitions of the quotient, which in turn aids the combinatorial-topological translation. For each family $\mathcal{G}$ of finite structures we define the class $[\![\mathcal{G}]\!]$ of compact metrizable structures for which such combinatorial-topological bridge holds. We use such notion to give conditions under which there is an approximately projectively homogeneous element in $[\![\mathcal{G}]\!]$.

Many proofs in the domain of projective Fraïssé theory are carried out in a context dependent fashion and have thus far eluded clean generalizations. A reason is to be found in the lack of a clear understanding of which spaces are amenable to be studied with projective Fraïssé limits. We give partial results in this direction in Sections 1.5 and 1.6, and Chapter 2. First we recontextualize a result by Panagiotopoulus [Pan16] which shows that in a more powerful setting all compact metrizable spaces are closely approximated by projective Fraïssé limits. Then we show that an analogous result does not hold in the original, less expressive, setting: all closed manifolds of dimension greater than one, as well as the Hilbert cube $[0,1]^{\mathbb{N}}$, cannot be closely approximated.

Incidentally, it has been recently proved in [GTZ19] that the UMFs of the groups of homeomorphism of the above spaces are not metrizable.

In Chapter 2, which is based on [BC17], we concentrate on the question of which compact metrizable spaces appear as domains of quotients of projective Fraïssé limits. We note that, if we admit infinite first order languages, then every compact metrizable space can be obtained as a quotient of a projective Fraïssé limit. We then restrict our attention to finite languages and prove, in Section 2.2, that the class of spaces that can be obtained as quotients of projective Fraïssé limits is closed under finite disjoint unions, finite products, and particular quotients satisfying some extra technical conditions. Section 2.3 presents some examples: after showing that arcs can be obtained as quotients of projective Fraïssé limits in a finite language, the results of Section 2.2 allow us to extend this property to hypercubes and graphs.

Chapters 3 and 4 are partially based on [BC20]. Using projective Fraïssé theory, we introduce and begin the study of a new class of topological spaces, which we call *fences*. These are the compact metrizable spaces whose connected components are either points or arcs. Among them, we define the subclass of *smooth fences* and characterize them as those fences admitting an embedding in $2^{\mathbb{N}} \times [0, 1]$.

In Chapter 3 we focus on a family $\mathcal{F}$ of structures — finite partial orders whose Hasse diagram is a forest — which we show (Theorem 3.1.6) is projective Fraïssé; its limit $\mathbb{F}$ admits a quotient $\mathbb{F}/R^{\mathbb{F}}$ which is a smooth fence. This space does not seem to appear in the literature, and we call it the *Fraïssé fence*. Chapter 4 is devoted to its study. We isolate a cofinal subclass $\mathcal{F}_0$ of $\mathcal{F}$ and we show that $[\![\mathcal{F}_0]\!]$ is the class of smooth fences (Theorem 3.4.1 and Corollary 3.4.4). We exploit the bridge between the combinatorial world and the topological one, which this result creates, to obtain a characterization (Theorem 4.1.2) of the Fraïssé fence by isolating a topological property which yields the amalgamation property for $\mathcal{F}_0$.

Fences, some of their properties, and the techniques we use, have analogs in the theory of fans. A *fan* is an arcwise connected and hereditarily unicoherent compact space that has at most one ramification point. A fan with ramification point t is *smooth* if for any sequence $(x_n)_{n\in\mathbb{N}}$ converging to x, the sequence $([t, x_n])_{n\in\mathbb{N}}$ of arcs connecting t to x_n converges to $[t, x]$. Smooth fans were introduced in [Cha67] and have been extensively studied in continuum theory. A *Lelek fan* is a smooth fan with a dense set of endpoints. Such a fan was first constructed in [Lel60] and was later proved to be unique up to homeomorphism in [BO90] and [Cha89].

In the series of papers [BK15, BK17, BK19], Bartošová and Kwiatkowska study the Lelek fan, and the dynamics of its homeomorphism group, via projective Fraïssé theory. By proving a novel structural Ramsey theorem and dualizing the methods from [KPT05], they prove that the UMF of its homeomorphisms group is metrizable and characterize it as the subspace of connected maximal chains of closed sets of the Lelek fan whose base point is the ramification point of the fan.

Besides the fact that both can be obtained as quotients of projective Fraïssé limits of some class of ordered structures, the Fraïssé fence and the Lelek fan share several other features:

- Both are as homogeneous as possible, namely they are $1/3$-homogeneous (see [AHPJ17] for the Lelek fan and Corollary 4.2.6 for the Fraïssé fence).

- Both are universal in the respective classes with respect to embeddings that preserve endpoints (see [DvM10] for the Lelek fan and Theorem 4.3.1 for the Fraïssé fence).

- For both, the set of endpoints is dense (see Proposition 4.4.4 for the Fraïssé fence). In fact, the Lelek fan is defined as the unique smooth fan with a dense set of endpoints; the Fraïssé fence too has a characterization in terms of denseness of endpoints (see Theorem 4.1.2).

- The set of endpoints of the Lelek fan is homeomorphic to the complete Erdős space ([KOT96]), a homogeneous, almost zero-dimensional, 1-dimensional space; the complete Erdős space is cohesive, that is, every point has a neighborhood which does not contain any nonempty clopen subset. Among the subspaces of the set of endpoints of the Fraïssé fence there is a homogeneous, almost zero-dimensional, 1-dimensional space $\mathfrak{M}$ which is not cohesive (Theorem 4.4.7(iv)).

A space with the properties mentioned for $\mathfrak{M}$ was constructed in [Dij06] as a counterexample to a question by Dijkstra and van Mill. This raises the question of whether the two examples are homeomorphic and whether they can be regarded as a non-cohesive analog of the complete Erdős space.

We conclude Chapter 4 by studying some dynamical properties of the Fraïssé fence, namely approximate projective homogeneity and the existence of a dense conjugacy class. Questions regarding the UMF of its group of automorphisms are left for further research.

Chapter 1 contains notions and results which are of use for the rest of the dissertation. Chapter 2 is independent of Chapters 3 and 4, which should be read sequentially. A good reference for basic model theory is [EFT94], one for general topology is [Eng89], and for descriptive set theory [Kec95].

Notation and conventions

Throughout this thesis, by *order* we mean partial order. We specify *total* (or *linear*) when needed. Given an order $(A, \leq)$ on a set A, a *chain* is a subset of A which is linearly ordered by $\leq$. A chain is *maximal* if it cannot be properly extended to a chain.

Let X be a topological space. If A is a subset of X, then $\operatorname{int}_X(A), \operatorname{cl}_X(A), \partial_X(A)$ denote the interior, closure, and boundary of A in X, respectively. We drop the subscript whenever the ambient space is clear from context. A closed set is regular if it coincides with the closure of its interior. We denote by $\mathcal{K}(X) = \{K \subseteq X \mid K \text{ compact}\}$ the space of compact subsets of X, with the Vietoris topology. This is the topology generated by the sets $\{K \in \mathcal{K}(X) \mid K \subseteq O\}$ and $\{K \in \mathcal{K}(X) \mid \forall i < n\ K \cap O_i \neq \emptyset\}$, for $n \in \mathbb{N}$ and $O, O_0, \ldots, O_{n-1}$ varying among the open subsets of X. If X is compact metrizable, so is $\mathcal{K}(X)$. Let $\operatorname{Homeo}(X)$ denote the group of homeomorphisms of X, which we endow with the compact-open topology. This is the topology whose basic open sets are $\{f \in \operatorname{Homeo}(X) \mid f[K] \subseteq O\}$, for K ranging among compact subsets of X and O among the open ones. If X is compact and d is a metric on X, then $d_{\sup}(f, g) = \sup_{x \in X} d(f(x), g(x))$ is a metric on $\operatorname{Homeo}(X)$. If $f : X \to Y$ is a function and $n \in \mathbb{N}$, we denote by $f^{(n)} : X^n \to Y^n$ the map $(x_0, \ldots, x_{n-1}) \mapsto (f(x_0), \ldots, f(x_{n-1}))$. By *mesh* of a covering of a metric space, we indicate the supremum of the diameters of its elements. When we talk about dimension, we mean the inductive dimension.

We collect here the definitions of some basic topological concepts we need.

- A space is *almost zero-dimensional* if each point has a neighborhood basis consisting of closed sets that are intersection of clopen sets.
- A space is *strongly σ-complete* if it is the union of countably many closed and completely metrizable subspaces.
- A space is X *cohesive* if each point has a neighborhood which does not contain any nonempty clopen subset of X.
- The *quasi-component* of a point is the intersection of all its clopen neighborhoods. A space is *totally separated* if the quasi-component of each point is a singleton.
- A space is *n-homogeneous* if for every two sets of n points there is a homeomorphism sending one onto the other. It is *ω-homogeneous* if it is n-homogeneous for each $n > 0$.
- A space X is *$1/n$-homogeneous* if the action of $\operatorname{Homeo}(X)$ on X has exactly n orbits.
- A space is *h-homogeneous* if it is homeomorphic to each of its nonempty clopen subsets.

Chapter 1

Projective Fraïssé theory

1.1 Compact metrizable structures

Let a relational first order language $\mathcal{L}$ be given. A *compact metrizable $\mathcal{L}$-structure* [1] is a compact metrizable space that is also an $\mathcal{L}$-structure such that the interpretations of the relation symbols are closed sets. In particular, the topology on finite topological $\mathcal{L}$-structures is discrete. We will usually suppress the words "compact metrizable" when referring to finite compact metrizable $\mathcal{L}$-structures.

An *epimorphism* between compact metrizable $\mathcal{L}$-structures A, B is a continuous surjection $\varphi : A \to B$ such that $r^B = \varphi^{(n)}[r^A]$, for every n-ary relation symbol $r \in \mathcal{L}$. Notice that epimorphisms are closed, since the domain is compact. An *isomorphism* is a bijective epimorphism, so in particular it is a homeomorphism between the supports. An isomorphism of A onto A is an *automorphism* and we denote by $\mathrm{Aut}(A)$ the group of automorphisms of A, with the topology inherited by $\mathrm{Homeo}(A)$. An epimorphism $\varphi : A \to B$ *refines* a covering $\mathcal{U}$ of A if the preimage of any point of B is included in some element of $\mathcal{U}$. If $\mathcal{G}, \mathcal{G}'$ are families of compact metrizable structures such that $\mathcal{G}' \subseteq \mathcal{G}$ and for all $A \in \mathcal{G}$ there exist $B \in \mathcal{G}'$ and an epimorphism $\varphi : B \to A$, we say that $\mathcal{G}'$ is *cofinal* in $\mathcal{G}$.

A family $\mathcal{G}$ of compact metrizable $\mathcal{L}$-structures is a *projective Fraïssé family* if the following properties hold:

(JPP) (joint projection property) for every $A, B \in \mathcal{G}$ there are $C \in \mathcal{G}$ and epimorphisms $C \to A$, $C \to B$;

(AP) (amalgamation property) for every $A, B, C \in \mathcal{G}$ and epimorphisms $\varphi_1 : B \to A$, $\varphi_2 : C \to A$ there are $D \in \mathcal{G}$ and epimorphisms $\psi_1 : D \to B$, $\psi_2 : D \to C$ such that $\varphi_1\psi_1 = \varphi_2\psi_2$.

[1] Compact metrizable structures were first defined in [RZ18]. In [IS06], where projective Fraïssé limits were introduced, the theory was developed for *topological $\mathcal{L}$-structures*, which in our notation are zero-dimensional compact metrizable $\mathcal{L}$-structures.

Given a family $\mathcal{G}$ of compact metrizable $\mathcal{L}$-structures, a zero-dimensional compact metrizable $\mathcal{L}$-structure $\mathbb{L}$ is a *projective Fraïssé limit* of $\mathcal{G}$ if the following hold:

(L1) (projective universality) for every $A \in \mathcal{G}$ there is some epimorphism $\mathbb{L} \to A$;

(L2) for any clopen covering $\mathcal{U}$ of $\mathbb{L}$ there are $A \in \mathcal{G}$ and an epimorphism $\mathbb{L} \to A$ refining $\mathcal{U}$;

(L3) (projective ultrahomogeneity) for every $A \in \mathcal{G}$ and epimorphisms $\varphi_1, \varphi_2 : \mathbb{L} \to A$ there exists an automorphism $\psi \in \mathrm{Aut}(\mathbb{L})$ such that $\varphi_2 = \varphi_1 \psi$.

Note that in the original definition of a projective Fraïssé limit in [IS06] item (L2) was replaced by a different but equivalent property.

If $\mathcal{G}$ is a projective Fraïssé family of finite $\mathcal{L}$-structures and $\mathbb{L}$ satisfies (L1) and (L2), then (L3) holds if and only if the following extension property holds:

(L3′) for any $A, B \in \mathcal{G}$ and epimorphisms $\varphi : B \to A$, $\psi : \mathbb{L} \to A$ there exists an epimorphism $\chi : \mathbb{L} \to B$ such that $\varphi\chi = \psi$.

The proof is the same as in [Pan16, Lemma 3].

In [IS06] it is proved that every nonempty, at most countable (up to isomorphism), projective Fraïssé family of finite $\mathcal{L}$-structures has a projective Fraïssé limit, which is unique up to isomorphism.

If $\mathcal{G}$ is a class of compact metrizable $\mathcal{L}$-structures, a *projective sequence* in $\mathcal{G}$ is a sequence $(A_n, \varphi_n^m)_{n\in\mathbb{N}, m\geq n}$, where:

- $A_n \in \mathcal{G}$;
- $\varphi_n^{n+1} : A_{n+1} \to A_n$ is an epimorphism, for each $n \in \mathbb{N}$;
- $\varphi_n^m = \varphi_n^{n+1} \cdots \varphi_{m-1}^m : A_m \to A_n$ for $n < m$, and $\varphi_n^n : A_n \to A_n$ is the identity.

A *projective limit* for such a sequence is a compact metrizable $\mathcal{L}$-structure $\mathbb{A}$, whose universe is $\mathbb{A} = \{u \in \prod_{n\in\mathbb{N}} A_n \mid \forall n \in \mathbb{N}\ u(n) = \varphi_n^{n+1}(u(n+1))\}$ and such that $r^{\mathbb{A}}(u_0, \ldots, u_{j-1}) \Leftrightarrow \forall n \in \mathbb{N}\ r^{A_n}(u_0(n), \ldots, u_{j-1}(n))$, for every j-ary relation symbol $r \in \mathcal{L}$. We denote by $\varphi_n : \mathbb{A} \to A_n$ the n-th projection map: this is an epimorphism. Notice that the projective limit of a sequence of finite $\mathcal{L}$-structures is zero-dimensional.

A *fundamental sequence* for $\mathcal{G}$ is a projective sequence (A_n, φ_n^m) such that the following properties hold:

(F1) $\{A_n\}_{n\in\mathbb{N}}$ is cofinal in $\mathcal{G}$;

(F2) for any n, any $A, B \in \mathcal{G}$ and any epimorphisms $\theta_1 : B \to A$, $\theta_2 : A_n \to A$, there exist $m \geq n$ and an epimorphism $\psi : A_m \to B$ such that $\theta_1 \psi = \theta_2 \varphi_n^m$.

To study projective Fraïssé limits it is enough to consider fundamental sequences, due to the following fact whose details can be found in [Cam10].

Proposition 1.1.1. *Let $\mathcal{G}$ be a nonempty, at most countable (up to isomorphism) family of finite $\mathcal{L}$-structures and $\mathcal{G}_0$ be cofinal in $\mathcal{G}$. Then the following are equivalent.*

1. *$\mathcal{G}$ is a projective Fraïssé family;*
2. *$\mathcal{G}$ has a projective Fraïssé limit;*
3. *$\mathcal{G}$ has a fundamental sequence.*

Moreover, in this case $\mathcal{G}_0$ is a projective Fraïssé family and the projective Fraïssé limits of $\mathcal{G}_0, \mathcal{G}$, and of its fundamental sequence coincide. A projective Fraïssé limit for them is the projective limit of the fundamental sequence.

If $\mathcal{G}$ is a projective Fraïssé family, one can check whether a given projective sequence is fundamental for $\mathcal{G}$ with the following.

Proposition 1.1.2. *Let $\mathcal{G}$ be a projective Fraïssé family of compact metrizable $\mathcal{L}$-structures. Let (A_n, φ_n^m) be a projective sequence in $\mathcal{G}$. Assume that for each $A \in \mathcal{G}$, $n \in \mathbb{N}$, and epimorphism $\theta : A \to A_n$, there exist $m \geq n$ and an epimorphism $\psi : A_m \to A$ such that $\theta\psi = \varphi_n^m$. Then (A_n, φ_n^m) is a fundamental sequence for $\mathcal{G}$.*

Proof. (F1) Let $A \in \mathcal{G}$, by (JPP) there exist $A' \in \mathcal{G}$, and epimorphisms $\varphi : A' \to A$ and $\varphi' : A' \to A_0$. By hypothesis there are n and an epimorphism $\theta : A_n \to A'$ such that $\varphi'\theta = \varphi_0^n$. Then $\varphi\theta$ is an epimorphism $A_n \to A$, as wished.

(F2) Let $A, B \in \mathcal{G}$ and epimorphisms $\theta_1 : B \to A$, $\theta_2 : A_n \to A$. By (AP) there exist $C \in \mathcal{G}$ and epimorphisms $\rho_1 : C \to B$ and $\rho_2 : C \to A_n$ such that $\theta_1\rho_1 = \theta_2\rho_2$. By hypothesis, there exist $m \geq n$ and an epimorphism $\psi' : A_m \to C$ such that $\rho_2\psi' = \varphi_n^m$. Then $\psi = \rho_1\psi' : A_m \to B$ is such that $\theta_1\psi = \theta_2\varphi_n^m$. $\square$

Notice that the converse of Proposition 1.1.2 holds as well.

1.2 Fine projective sequences

In the sequel, whenever we denote a language with a subscript, like in $\mathcal{L}_R$, we mean that the language contains a distinguished binary relation symbol represented in the subscript. The interpretation of R in a compact metrizable $\mathcal{L}_R$-structure is expected to be reflexive and symmetric. These properties are preserved under projective limits. A compact metrizable $\mathcal{L}_R$-structure A in which the interpretation of R is the identity is called an $\mathcal{L}_R$-*quotient*. An $\mathcal{L}_R$-*prespace* is any zero dimensional compact metrizable $\mathcal{L}_R$-structure A in which the interpretation of R is transitive, that is, an equivalence relation. In such case, let $p : A \to A/R^A$ denote the quotient map, and let A/R^A be endowed with the $\mathcal{L}_R$-structure where $r^{A/R^A} = p^{(n)}(r^A)$, for any $r \in \mathcal{L}_R \setminus \{R\}$ of arity n, and R^{A/R^A} is the identity. Since R^A is a closed equivalence relation, p is a closed map, and therefore each r^{A/R^A} is also closed, so A/R^A is a $\mathcal{L}_R$-quotient and p is an epimorphism. In this case, we say that A is a prespace of A/R^A.

Remark 1.2.1. In the above setting, let $\alpha \in \mathrm{Aut}(A)$ be an automorphism of the prespace. Then there is a unique $\alpha^* \in \mathrm{Aut}(A/R^A)$ such that $p\alpha = \alpha^* p$. The map $p^* : \mathrm{Aut}(A) \to \mathrm{Aut}(A/R^A)$ which sends α to α^* is a continuous homomorphism of topological groups. As we see in Section 1.3, if p is irreducible, then p^* is an embedding. Particular relevance in the literature is given to the case in which p^* has dense image. Indeed, in this case it is possible to transfer some results regarding the dynamics of A to corresponding results for A/R^A, see Corollary 4.5.5 or [BK19, Theorem 5.3], for example.

Definition 1.2.2. A projective sequence (A_n, φ_n^m) of finite $\mathcal{L}_R$-structures and epimorphisms is *fine* whenever its projective limit is an $\mathcal{L}_R$-prespace. If (A_n, φ_n^m) is a fine projective sequence in $\mathcal{L}_R$ with projective limit $\mathbb{A}$ and X is a compact metrizable $\mathcal{L}_R$-quotient isomorphic to $\mathbb{A}/R^{\mathbb{A}}$, we say that (A_n, φ_n^m) *approximates* X.

Given a reflexive graph (that is, a reflexive and symmetric relation) R on some set, denote by d_R the distance on the graph, where $d_R(a,b) = \infty$ if a, b belong to distinct connected components of the graph. Note that if R, S are reflexive graphs and φ is a function between them such that $x \, R \, y \Rightarrow \varphi(x) \, S \, \varphi(y)$ for all x, y, then the inequality $d_S(\varphi(x), \varphi(y)) \leq d_R(x,y)$ holds for every x, y.

We can determine whether a sequence is fine by checking that the R-distance of points which are not R-related tends to infinity. More precisely:

Lemma 1.2.3. *Let (A_n, φ_n^m) be a projective sequence of finite $\mathcal{L}_R$-structures, with projective limit $\mathbb{A}$. Assume that R^{A_n} is reflexive and symmetric for every $n \in \mathbb{N}$. The projective sequence is fine if and only if for all $n \in \mathbb{N}$ and $a, b \in A_n$ with $d_{R^{A_n}}(a,b) = 2$, there is $m > n$ such that if $a' \in (\varphi_n^m)^{-1}(a), b' \in (\varphi_n^m)^{-1}(b)$ then $d_{R^{A_m}}(a', b') \geq 3$.*

Proof. Let $a, b \in A_n$ with $d_{R^{A_n}}(a,b) = 2$, say $a \, R^{A_n} \, c \, R^{A_n} \, b$. If for each $m > n$ there are $a_m \in (\varphi_n^m)^{-1}(a), b_m \in (\varphi_n^m)^{-1}(b)$ with $d_{R^{A_m}}(a_m, b_m) = 2$, say $a_m \, R^{A_m} \, c_m \, R^{A_m} \, b_m$, let

$$x_m \in \varphi_m^{-1}(a_m), \quad y_m \in \varphi_m^{-1}(b_m), \quad z_m, z'_m \in \varphi_m^{-1}(c_m),$$

with $x_m \, R^{\mathbb{A}} \, z_m, z'_m \, R^{\mathbb{A}} \, y_m$. Passing to a suitable subsequence, let

$$x = \lim_{h \to \infty} x_{m_h}, \quad y = \lim_{h \to \infty} y_{m_h}, \quad z = \lim_{h \to \infty} z_{m_h} = \lim_{h \to \infty} z'_{m_h},$$

so that $x \, R^{\mathbb{A}} \, z \, R^{\mathbb{A}} \, y$. However, x, y are not $R^{\mathbb{A}}$-related (otherwise $a \, R^{A_n} \, b$), so (A_n, φ_n^m) is not fine.

On the other hand, if (A_n, φ_n^m) is not fine there are $x, y \in \mathbb{A}$ such that $d_{R^{\mathbb{A}}}(x,y) = 2$, say $x \, R^{\mathbb{A}} \, z \, R^{\mathbb{A}} \, y$, for x, y, z distinct points. There is $n \in \mathbb{N}$ such that for all $m \geq n$ the points $\varphi_m(x), \varphi_m(y), \varphi_m(z)$ are distinct and $(\varphi_m(x), \varphi_m(y)) \notin R^{A_m}$, so $d_{R^{A_m}}(\varphi_m(x), \varphi_m(y)) = 2$. Therefore the property does not hold for $\varphi_n(x), \varphi_n(y)$. □

Definition 1.2.4. Let A be a compact metrizable $\mathcal{L}_R$-structure and $B \subseteq A$. We say B is *R-connected* if for any two clopen sets $U, U' \subseteq A$ such that $U \cap B, U' \cap B$ partition B, there are $x \in U \cap B, x' \in U' \cap B$ such that $x \; R^A \; x'$.

Notice that if A is a finite $\mathcal{L}_R$-structure, R-connectedness coincides with the usual notion of connectedness for the graph R^A, and if A is a $\mathcal{L}_R$-quotient, it is the usual topological connectedness.

Lemma 1.2.5. *Let $\varphi : A' \to A$ be an epimorphism between compact metrizable $\mathcal{L}_R$-structures. Then the image $\varphi[B]$ of a closed R-connected subset $B \subseteq A'$ is closed and R-connected.*

Proof. The set $\varphi[B]$ is closed as φ is a closed map. Suppose that U, U' are clopen subsets of A such that $U \cap \varphi[B], U' \cap \varphi[B]$ partition $\varphi[B]$. Then $\varphi^{-1}(U), \varphi^{-1}(U')$ are clopen subsets of A' such that $\varphi^{-1}(U) \cap B, \varphi^{-1}(U') \cap B$ partition B. By the assumption, there are $u \in \varphi^{-1}(U) \cap B, u' \in \varphi^{-1}(U') \cap B$ with $u R^{A'} u'$, and since φ is an epimorphism, $\varphi(u) \; R^A \; \varphi(u')$. So $\varphi[B]$ is R-connected. $\square$

Definition 1.2.6. Let A be a finite $\mathcal{L}_R$-structure and $\mathcal{C}$ be a cover of a compact metrizable $\mathcal{L}_R$-quotient X. We say that $\mathcal{C}$ is *A-like* whenever there is a bijection between A and $\mathcal{C}$, denoted by $a \mapsto C_a$, such that:

(A0) for each $a \in A$, $C_a \setminus \bigcup_{a' \neq a} C_{a'} \neq \emptyset$;

(A1) $a \; R^A \; a'$ if and only if $C_a \cap C_{a'} \neq \emptyset$;

(A2) for $r \in \mathcal{L}_R \setminus \{R\}$ of arity n, if $(x_0, \dots, x_{n-1}) \in r^X$, then there is $(a_0, \dots, a_{n-1}) \in r^A$ such that $x_i \in C_{a_i}$, for each $i < n$;

(A3) for $r \in \mathcal{L}_R \setminus \{R\}$ of arity n, if $(a_0, \dots, a_{n-1}) \in r^A$, then there is $(x_0, \dots, x_{n-1}) \in r^X$ such that
$$x_i \in C_{a_i} \setminus \bigcup_{a \neq a_i} C_a,$$
for each $i < n$.

If $\mathcal{G}$ is a family of finite $\mathcal{L}_R$-structures we say that a cover $\mathcal{C}$ of X is $\mathcal{G}$-like if there is $A \in \mathcal{G}$ such that $\mathcal{C}$ is A-like.

We often treat a $\mathcal{G}$-like cover as an element of $\mathcal{G}$, by endowing it with the discrete topology and the $\mathcal{L}_R$-structure which makes $a \mapsto C_a$ an isomorphism.

Lemma 1.2.7. *Let X', X be compact metrizable $\mathcal{L}_R$-quotients, $\varphi : X' \to X$ be an epimorphism, and A be a finite $\mathcal{L}_R$-structure. If $\mathcal{C}$ is an A-like cover of X, then $\varphi^{-1}\mathcal{C} = \{\varphi^{-1}(C_a) \mid a \in A\}$ is an A-like cover of X'.*

Proof. Properties (A0), (A1) hold by surjectivity.

Fix $r \in \mathcal{L}_R \setminus \{R\}$ of arity n. If $(x'_0, \ldots, x'_{n-1}) \in r^{X'}$, then $\varphi^{(n)}(x'_0, \ldots, x'_{n-1}) \in r^X$, so there are $(a_0, \ldots, a_{n-1}) \in r^A$ such that $\varphi(x'_i) \in C_{a_i}$, that is, $x'_i \in \varphi^{-1}(C_{a_i})$, for each $i < n$. This takes care of (A2).

To check that (A3) holds, let $(a_0, \ldots, a_{n-1}) \in r^A$. Then there is $(x_0, \ldots, x_{n-1}) \in r^X$ such that $x_i \in C_{a_i} \setminus \bigcup_{a \neq a_i} C_a$, for each $i < n$. Since φ is an epimorphism, there is $(x'_0, \ldots, x'_{n-1}) \in r^{X'}$ such that $\varphi(x'_i) = x_i$, for $i < n$. It follows that $x'_0 \in \varphi^{-1}(C_{a_i}) \setminus \bigcup_{a \neq a_i} \varphi^{-1}(C_a)$. □

For the remainder of the section we fix a fine projective sequence of finite $\mathcal{L}_R$-structures (A_n, φ_n^m) with projective limit $\mathbb{A}$ and with quotient map $p : \mathbb{A} \to \mathbb{A}/R^{\mathbb{A}}$.

If $\varphi : \mathbb{A} \to A$ is an epimorphism onto a finite $\mathcal{L}_R$-structure A and $a \in A$, we let

$$[\![a]\!]_\varphi = p\big[\varphi^{-1}(a)\big], \qquad [\![A]\!]_\varphi = \{[\![a]\!]_\varphi \mid a \in A\}.$$

Remark 1.2.8. Notice that the cover $[\![A]\!]_\varphi$ of $\mathbb{A}/R^{\mathbb{A}}$ is not necessarily A-like, because (A0), (A3) may fail. In Section 1.3 we give conditions under which they hold.

Lemma 1.2.9.

1. *The mesh of the sequence* $\left(\{\varphi_n^{-1}(a) \mid a \in A_n\}\right)_{n \in \mathbb{N}}$ *tends to* 0. *In particular, the sets* $\varphi_n^{-1}(a)$ *for* $n \in \mathbb{N}, a \in A_n$ *form a basis for the topology of* $\mathbb{A}$.

2. *The mesh of the sequence* $\left([\![A_n]\!]_{\varphi_n}\right)_{n \in \mathbb{N}}$ *tends to* 0.

Proof. (1) Suppose that there is $\varepsilon > 0$ such that for infinitely many $n \in \mathbb{N}$, there is $a_n \in A_n$ with $\operatorname{diam}(\varphi_n^{-1}(a_n)) \geq \varepsilon$. Consider the forest $T = \{\varphi_{n'}^n(a_n) \mid n' < n\}$, so that $\operatorname{diam}(\varphi_n^{-1}(b)) \geq \varepsilon$ for every b in the forest, if $b \in A_n$. Let $u = (b_0, b_1, \ldots) \in \mathbb{A}$ be an infinite branch in T. Since

$$n < n' \Rightarrow \varphi_{n'}^{-1}(b_{n'}) \subseteq \varphi_n^{-1}(b_n)$$

it follows that the sequence $\varphi_n^{-1}(b_n)$ converges in $\mathcal{K}(\mathbb{A}/R^{\mathbb{A}})$ to $K = \bigcap_{n \in \mathbb{N}} \varphi_n^{-1}(b_n)$ with $\operatorname{diam}(K) \geq \varepsilon$. But $\bigcap_{n \in \mathbb{N}} \varphi_n^{-1}(b_n) = \{u\}$, a contradiction.

(2) By (1) and the fact that function p is uniformly continuous. □

Lemma 1.2.10. *If* $B_n \subseteq A_n$, *for* $n \in \mathbb{N}$, *are* R*-connected subsets and* $(\varphi_n^{-1}(B_n))_{n \in \mathbb{N}}$ *converges in* $\mathcal{K}(\mathbb{A})$ *to* K, *then* K *is* R*-connected.*

Proof. Let U, U' be clopen, nonempty subsets of $\mathbb{A}$, with some positive distance δ, such that $U \cap K, U' \cap K$ partition K. Consider the open neighborhood $O = \{C \in \mathcal{K}(\mathbb{A}) \mid C \subseteq U \cup U', C \cap U \neq \emptyset, C \cap U' \neq \emptyset\}$ of K in $\mathcal{K}(\mathbb{A})$. Let $n \in \mathbb{N}$ be such that $\varphi_n^{-1}(B_n) \in O$, and $\operatorname{diam}(\varphi_n^{-1}(a)) < \delta$ for each $a \in A_n$: such a n exists by Lemma 1.2.9. Then each $\varphi_n^{-1}(a)$

for $a \in B_n$ is either contained in U or in U', as the distance between the two clopen sets is greater than its diameter, and U, U' each contain at least one such set, since $\varphi_n^{-1}(B_n)$ has nonempty intersection with both U and U'. It follows that $\varphi_n[U] \cap B_n, \varphi_n[U'] \cap B_n$ partition B_n. But B_n is R-connected, so there are $a \in B_n \cap \varphi_n[U], a' \in B_n \cap \varphi_n[U']$ such that $a \, R^{A_n} \, a'$, and thus there exist $x \in \varphi_n^{-1}(a) \subseteq U, x' \in \varphi_n^{-1}(a') \subseteq U'$ such that $x \, R^{\mathbb{A}} \, x'$. So K is R-connected. □

Corollary 1.2.11. *If $B_n \subseteq A_n$ are R-connected subsets and $(\bigcup_{a \in B_n} [\![a]\!]_{\varphi_n})_{n \in \mathbb{N}}$ converges in $\mathcal{K}(\mathbb{A}/R^{\mathbb{A}})$ to some K, then K is connected.*

Proof. Let n_k be an increasing sequence of natural numbers such that $\varphi_{n_k}^{-1}(B_{n_k})$ converges in $\mathcal{K}(\mathbb{A})$, say $\lim_{k\to\infty} \varphi_{n_k}^{-1}(B_{n_k}) = L$. Then

$$\lim_{n\to\infty} \bigcup_{a \in B_n} [\![a]\!]_{\varphi_n} = \lim_{n\to\infty} p[\varphi_n^{-1}(B_n)] = \lim_{k\to\infty} p[\varphi_{n_k}^{-1}(B_{n_k})] = p[L],$$

whence $K = p[L]$. Now apply Lemmas 1.2.5 and 1.2.10. □

1.3 Irreducible functions and regular quasi-partitions

Given topological spaces X, Y, a continuous map $f : X \to Y$ is *irreducible* if $f[K] \neq Y$ for all proper closed subsets $K \subset X$.

We recall some basic results on irreducible closed surjective maps between compact metrizable spaces [2], whose proofs can be found in [AP84]. Let $f : X \to Y$ be such a map. Given $A \subseteq X$, let $f^{\#}(A) = \{y \in Y \mid f^{-1}(y) \subseteq A\}$. If $O \subseteq Y$ is an open set, then $f^{\#}(O)$ is open and $f^{-1}(f^{\#}(O))$ is dense in O. If $C \subseteq X$ is a regular closed set, then $C = \mathrm{cl}(f^{-1}(f^{\#}(\mathrm{int}(C))))$, and $f[C] = \mathrm{cl}(f^{\#}(\mathrm{int}(C)))$, so in particular the image of a regular closed set is regular. The preimage of any point by f is either an isolated point or has empty interior. If C, C' are regular closed and $f[C] = f[C']$ then $C = C'$; if $\mathrm{int}(C \cap C') = \emptyset$ then $\mathrm{int}(f[C] \cap f[C']) = \emptyset$.

Definition 1.3.1. A covering $\mathcal{C}$ of a topological space is a *regular quasi-partition* if the elements of $\mathcal{C}$ are nonempty, regular closed sets and $\forall C, C' \in \mathcal{C}$ $(C \neq C' \Rightarrow C \cap C' \subseteq \partial(C) \cap \partial(C'))$.

Remark 1.3.2. If $\mathcal{C}$ is a regular quasi-partition, then $C \setminus \bigcup_{C' \neq C} C' = \mathrm{int}\, C \neq \emptyset$. It follows that if $\mathcal{C}'$ is a regular quasi-partition which refines $\mathcal{C}$, then for each $C' \in \mathcal{C}'$ there is a unique $C \in \mathcal{C}$ such that $C' \subseteq C$. On the other hand, for each $C \in \mathcal{C}$, there is $C' \in \mathcal{C}'$, $C' \subseteq C$. The refinement therefore gives rise to a surjective function $\mathcal{C}' \to \mathcal{C}$.

Lemma 1.3.3. *If X, Y are compact metrizable spaces and $f : X \to Y$ is an irreducible closed surjective map, then the image $f\mathcal{C} = \{f[C] \mid C \in \mathcal{C}\}$ of a regular quasi-partition $\mathcal{C}$ of X is a regular quasi-partition of Y, and the map $C \mapsto f[C]$ is a bijection between $\mathcal{C}$ and $f\mathcal{C}$.*

[2] in this context the notions of irreducible, almost 1-to-1, and highly proximal coincide.

Proof. The fact that $C \mapsto f[C]$ is a bijection is one of the basic properties of irreducible closed surjective maps between compact metrizable spaces. The same for the fact that each $f[C]$ is a regular closed set.

Assume now that $C, C' \in \mathcal{C}$, and let $x \in f[C] \cap f[C']$. We show that $x \notin \mathrm{int}(f[C])$, similarly $x \notin \mathrm{int}(f[C'])$. If toward contradiction $x \in \mathrm{int}(f[C])$, let O be open with $x \in O \subseteq f[C]$. Since $x \in f[C']$ and $f[C']$ is regular closed, there is $y \in O \cap \mathrm{int}(f[C'])$, so that there exists an open set V with $y \in V \subseteq f[C] \cap f[C']$. It follows that $\mathrm{int}(f[C] \cap f[C']) \neq \emptyset$, whence $\mathrm{int}(C \cap C') \neq \emptyset$ and then $\mathrm{int}(C) \cap \mathrm{int}(C') \neq \emptyset$, against $\mathcal{C}$ being a regular quasi-partition. $\square$

Fix again a fine projective sequence of finite $\mathcal{L}_R$-structures (A_n, φ_n^m) with projective limit $\mathbb{A}$ and with quotient map $p : \mathbb{A} \to \mathbb{A}/R^{\mathbb{A}}$.

Lemma 1.3.4. *The following are equivalent:*

1. *The set of points of $\mathbb{A}$ whose $R^{\mathbb{A}}$-equivalence class is a singleton is dense.*
2. *For each $n \in \mathbb{N}$ and $a \in A_n$ there are $m > n$ and $b \in A_m$ such that if $b' \, R^{A_m} \, b$ then $\varphi_n^m(b') = a$.*
3. *The quotient map $p : \mathbb{A} \to \mathbb{A}/R^{\mathbb{A}}$ is irreducible.*

Proof. Let M denote the set of points of $\mathbb{A}$ whose $R^{\mathbb{A}}$-equivalence class is a singleton. (1) $\Rightarrow$ (3). Let $K \subset \mathbb{A}$ be a proper closed subset. Then there is $x \in M \setminus K$, so that $p(x) \notin p[K]$. Thus p is irreducible.

(3) $\Rightarrow$ (2). Let $n \in \mathbb{N}$ and $a \in A_n$. By irreducibility of p,

$$O = p^{-1}(p^{\#}(\varphi_n^{-1}(a))) = \{x \in \mathbb{A} \mid [x]_{R^{\mathbb{A}}} \subseteq \varphi_n^{-1}(a)\}$$

is an open, nonempty, and $R^{\mathbb{A}}$-invariant set contained in $\varphi_n^{-1}(a)$. Let $m > n$ and $b \in A_m$ be such that $\varphi_m^{-1}(b) \subseteq O$, which exist since such sets are a basis for the topology on $\mathbb{A}$. If $b' \, R^{A_m} \, b$, there are $x \in \varphi_m^{-1}(b), x' \in \varphi_m^{-1}(b')$ such that $x \, R^{\mathbb{A}} \, x'$. But $x \in \varphi_m^{-1}(b) \subseteq O$, which is $R^{\mathbb{A}}$-invariant, so also $x' \in O$. It follows that $\varphi_n(x') = a$ and thus $\varphi_n^m(b') = a$, for $\varphi_n = \varphi_n^m \varphi_m$.

(2) $\Rightarrow$ (1). Since $\{\varphi_n^{-1}(a) \mid n \in \mathbb{N}, a \in A_n\}$ is a basis for the topology on $\mathbb{A}$ it suffices to fix $n \in \mathbb{N}$ and $a \in A_n$ and prove that there is $x \in M$ with $\varphi_n(x) = a$. We construct a sequence n_i and elements $b_i \in A_{n_i}$ by induction. Let $n_0 = n$ and $b_0 = a$. Given $b_i \in A_{n_i}$, by hypothesis there are $m > n_i$ and $b \in A_m$ such that whenever $b' \, R^{A_m} \, b$ it follows that $\varphi_{n_i}^m(b') = b_i$. Set $n_{i+1} = m$ and $b_{i+1} = b$. Thus $\varphi_{n_i}^{n_{i+1}}(b_{i+1}) = b_i$ for each i, so there exists $x \in \mathbb{A}$ such that $\varphi_{n_i}(x) = b_i$, for each $i \in \mathbb{N}$. In particular $\varphi_n(x) = a$. Let $y \, R^{\mathbb{A}} \, x$; if towards contradiction $y \neq x$ then there is $i \in \mathbb{N}$ such that $\varphi_{n_i}(y) \neq \varphi_{n_i}(x) = b_i$. But $\varphi_{n_{i+1}}(y) \, R^{A_{n_{i+1}}} \, \varphi_{n_{i+1}}(x) = b_{i+1}$, so $\varphi_{n_i}^{n_{i+1}} \varphi_{n_{i+1}}(y) = \varphi_{n_i}(y) = b_i$ by construction of b_{i+1}, a contradiction. $\square$

Prespaces whose quotient map is irreducible are central enough to our work that they merit a name.

Definition 1.3.5. If the quotient map $p : \mathbb{A} \to \mathbb{A}/R^{\mathbb{A}}$ is irreducible and X is a compact metrizable $\mathcal{L}_R$-structure isomorphic to $\mathbb{A}/R^{\mathbb{A}}$ we say that (A_n, φ_n^m) approximates X *closely*, and that $\mathbb{A}$ is a *close* $\mathcal{L}_R$-prespace.

Remark 1.3.6. In the above setting the homomorphism $p^* : \mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{A}/R^{\mathbb{A}})$ induced by the quotient map is an embedding. Indeed, it is enough to show that if $\alpha \in \mathrm{Aut}(\mathbb{A})$ is not the identity, then neither is α^*. So let $U \subseteq \mathbb{A}$ be a clopen set such that $\alpha[U] \cap U = \emptyset$. Since p is irreducible, $p[U], p[\alpha[U]]$ are regular closed sets whose intersection has empty interior. But $p[\alpha[U]] = \alpha^*[p[U]]$, so α^* is not the identity.

By Lemma 1.3.3 we have the following.

Proposition 1.3.7. *If $\mathbb{A}$ is a close $\mathcal{L}_R$-prespace, then $[\![A]\!]_\varphi$ is a regular quasi-partition of $\mathbb{A}/R^{\mathbb{A}}$ and the function*

$$a \in A \mapsto [\![a]\!]_\varphi \in [\![A]\!]_\varphi$$

is a bijection.

Lemma 1.3.8. *Suppose that $\mathbb{A}$ is a close $\mathcal{L}_R$-prespace. For every $n \in \mathbb{N}$, $a \in A_n$,*

$$\partial([\![a]\!]_{\varphi_n}) = \{x \in [\![a]\!]_{\varphi_n} \mid \exists a' \neq a, a' \, R^{A_n} \, a, x \in [\![a']\!]_{\varphi_n}\} = \{x \in [\![a]\!]_{\varphi_n} \mid \exists a' \neq a, x \in [\![a']\!]_{\varphi_n}\}.$$

Moreover, if p is at most 2-to-1 then for each x there are at most two $a \in A_n$ such that $x \in [\![a]\!]_{\varphi_n}$.

Proof. Let $x \in \partial([\![a]\!]_{\varphi_n})$, so that $x = p(u)$ for some $u \in \varphi_n^{-1}(a)$. As each $[\![a']\!]_{\varphi_n}$ is closed, this implies that there exists $a' \in A_n, a' \neq a$ such that $x \in [\![a']\!]_{\varphi_n}$, so that there is $v \in \varphi_n^{-1}(a')$ with $u \, R^{\mathbb{A}} \, v$; in turns, this entails that $a \, R^{A_n} \, a'$.

Let now $x \in [\![a]\!]_{\varphi_n}$, and assume that there exists $a' \in A_n$, with $a' \neq a, x \in [\![a']\!]_{\varphi_n}$. Since $[\![a]\!]_{\varphi_n} \cap [\![a']\!]_{\varphi_n} \subseteq \partial([\![a]\!]_{\varphi_n}) \cap \partial([\![a']\!]_{\varphi_n})$, it follows that $x \in \partial([\![a]\!]_{\varphi_n})$.

The last statement is a direct consequence of the definition of $[\![a]\!]_{\varphi_n}$. □

1.4 Suitable sequences

Definition 1.4.1. Let X be a compact metrizable $\mathcal{L}_R$-quotient, $\mathcal{G}$ be a family of finite $\mathcal{L}_R$-structures, and $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a sequence of covers of X. We say that $(\mathcal{C}_n)_{n\in\mathbb{N}}$ is a $\mathcal{G}$*-suitable* sequence of X if each $\mathcal{C}_n$ is a $\mathcal{G}$-like regular quasi-partition, $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$ for each n, and the mesh of $\mathcal{C}_n$ tends to 0.

The following is a combinatorial criterion which a fine sequence (A_n, φ_n^m) of finite $\mathcal{L}_R$-structures with limit $\mathbb{A}$ has to satisfy in order to give rise to a suitable sequence of $\mathbb{A}/R^{\mathbb{A}}$.

Proposition 1.4.2. *Suppose that for each $r \in \mathcal{L}_R \setminus \{R\}$ of arity ℓ, each $n \in \mathbb{N}$ and $(a_0, \ldots, a_{\ell-1}) \in r^{A_n}$, there are $m > n$ and $(b_0, \ldots, b_{\ell-1}) \in r^{A_m}$ such that for each $i < \ell$, whenever $b' \, R^{A_m} \, b_i$ then $\varphi_n^m(b') = a_i$. Then for each $n \in \mathbb{N}$, $[\![A_n]\!]_{\varphi_n}$ is a cover of $\mathbb{A}/R^{\mathbb{A}}$ which satisfies (A3) with respect to A_n. If furthermore $\mathbb{A}$ is a close prespace, then $[\![A_n]\!]_{\varphi_n}$ is A_n-like, and $([\![A_n]\!]_{\varphi_n})_{n\in\mathbb{N}}$ is a $\{A_n \mid n \in \mathbb{N}\}$-suitable sequence of regular quasi-partitions.*

Proof. Fix $n \in \mathbb{N}$ and let $(a_0, \ldots, a_{\ell-1}) \in r^{A_n}$. By hypothesis there are $m > n$ and $(b_0, \ldots, b_{\ell-1}) \in r^{A_m}$ such that for each $i < \ell$, whenever $b' \, R^{A_m} \, b_i$ then $\varphi_n^m(b') = a_i$. It follows that if $u \in \varphi_m^{-1}(b_i)$ and $u' \, R^{\mathbb{A}} \, u$, then $\varphi_m(u') = b_i$, so $[\![b_i]\!]_{\varphi_m} \subseteq [\![a_i]\!]_{\varphi_n} \setminus \bigcup_{a \neq a_i} [\![a]\!]_{\varphi_n}$, for $i < \ell$. Since φ_m is an epimorphism there is $(u_0, \ldots, u_{\ell-1}) \in r^{\mathbb{A}}$ such that $\varphi_m(u_i) = b_i$, for each $i < \ell$, so (A3) holds.

The rest of the proof follows from Lemma 1.2.9 and Proposition 1.3.7. □

We show, conversely, that each $\mathcal{G}$-suitable sequence gives rise to a fine a projective sequence of $\mathcal{G}$-structures.

Proposition 1.4.3. *Let X be a compact metrizable $\mathcal{L}_R$-quotient, $\mathcal{G}$ be a family of finite $\mathcal{L}_R$-structures, and $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a $\mathcal{G}$-suitable sequence of X. For each $m \geq n$, let $\chi_n^m : \mathcal{C}_m \to \mathcal{C}_n$ be the inclusion map, that is, $\chi_n^m(C) = C'$ if and only if $C \subseteq C'$. Then $(\mathcal{C}_n, \chi_n^m)$ is a fine projective sequence of structures from $\mathcal{G}$ closely approximating X such that $[\![C]\!]_{\chi_n} = C$, for any $n \in \mathbb{N}, C \in \mathcal{C}_n$.*

Proof. We prove that each χ_n^m is an epimorphism. By Remark 1.3.2 it is a surjective function. If $C, C' \in \mathcal{C}_m$ are such that $C \, R^{\mathcal{C}_m} \, C'$, then $C \cap C' \neq \emptyset$, so that $\chi_n^m(C) \cap \chi_n^m(C') \neq \emptyset$ and then $\chi_n^m(C) \, R^{\mathcal{C}_n} \, \chi_n^m(C')$. If $C, C' \in \mathcal{C}_n$ are such that $C \, R^{\mathcal{C}_n} \, C'$, then $C \cap C' \neq \emptyset$, so let $x \in C \cap C'$. Since, by the regularity of C, C', point x is in the closure of the interior both of C and C', there are $D, D' \in \mathcal{C}_m$ such that $x \in D \subseteq C, x \in D' \subseteq C'$, so that $D \, R^{\mathcal{C}_m} \, D'$.

Now let $r \in \mathcal{L}_R \setminus \{R\}$ be of arity ℓ. Assume $C_0, \ldots, C_{\ell-1} \in \mathcal{C}_m$, with $(C_0, \ldots, C_{\ell-1}) \in r^{\mathcal{C}_m}$. By (A3), there is $(x_0, \ldots, x_{\ell-1}) \in r^X$ such that $x_i \in \mathrm{int}(C_i)$, for each $i < \ell$. Then $x_i \in \mathrm{int}(\chi_n^m(C_i))$, for $i < \ell$, so $(\chi_n^m(C_0), \ldots, \chi_n^m(C_{\ell-1})) \in r^{\mathcal{C}_n}$ by (A2).

Finally, assume that $C_0, \ldots, C_{\ell-1} \in \mathcal{C}_n$, with $(C_0, \ldots, C_{\ell-1}) \in r^{\mathcal{C}_n}$. By (A3) there is $(x_0, \ldots, x_{\ell-1}) \in r^X$ such that $x_i \in \mathrm{int}(C_i)$, for each $i < \ell$. By (A2) there is $(C'_0, \ldots, C'_{\ell-1}) \in r^{\mathcal{C}_m}$ with $x_i \in C'_i$, for $i < \ell$. Fix $i < \ell$. Since $x_i \in \mathrm{int}(C_i) \cap C'_i$, and $\mathcal{C}_m, \mathcal{C}_n$ are regular quasi-partitions, it follows that $C'_i \subseteq C_i$, that is, $\chi_n^m(C'_i) = C_i$.

We prove that the sequence is fine. Let $\mathbb{X}$ be the projective limit of $(\mathcal{C}_n, \chi_n^m)$. Relation $R^{\mathbb{X}}$ is reflexive and symmetric, as all $R^{\mathcal{C}_n}$ are. Since the mesh of $(\mathcal{C}_n)$ tends to 0, Lemma 1.2.3 allows to conclude that the sequence is fine.

To check that the quotient map $p : \mathbb{X} \to \mathbb{X}/R^{\mathbb{X}}$ is irreducible, we apply Lemma 1.3.4 by showing that given $n \in \mathbb{N}, D \in \mathcal{C}_n$, the set $\chi_n^{-1}(D)$ contains a point whose $R^{\mathbb{X}}$-equivalence class is a singleton. Since $Q = \bigcap_{m\in\mathbb{N}} \bigcup_{C\in\mathcal{C}_m} \mathrm{int}(C)$ is dense in Y, let

$x \in Q \cap D$; then for each m there is exactly one $C_m \in \mathcal{C}_m$ to which x belongs, so $p^{-1}(x) = \{(C_m)_{m\in\mathbb{N}}\}$ is not $R^{\mathbb{Y}}$-related to any other point and $p^{-1}(x) \in \chi_n^{-1}(D)$.

Finally, we prove that $\mathbb{X}/R^{\mathbb{X}}$ is isomorphic to X. Since the mesh of the sequence $(\mathcal{C}_n)$ tends to 0, the function $q : \mathbb{X} \to X$ assigning to each $u \in \mathbb{X}$ the unique element of $\bigcap_{n\in\mathbb{N}} \chi_n(u)$ is well defined, and $q[\chi_n^{-1}(C)] = C$, for any $n \in \mathbb{N}$ and $C \in \mathcal{C}_n$. It is surjective since every member of $\mathcal{C}_n$ is covered by the members of $\mathcal{C}_{n+1}$ contained in it, and it is is continuous as the mesh of $(\mathcal{C}_n)$ goes to 0.

Let $r \in \mathcal{L}_R \setminus \{R\}$ have arity ℓ. If $(u_0, \dots, u_{\ell-1}) \in r^{\mathbb{X}}$ then $(\chi_n(u_0), \dots, \chi_n(u_{\ell-1})) \in r^{\mathcal{C}_n}$, for all $n \in \mathbb{N}$. It follows that for each $n \in \mathbb{N}$, there are $(x_0^n, \dots, x_{\ell-1}^n) \in r^X$, with $x_i^n \in \operatorname{int}(\chi_n(u_i))$, for $i < \ell$. By closure of r^X, it follows that $(q(u_0), \dots, q(u_{\ell-1})) \in r^X$.

On the other hand, let $(x_0, \dots, x_{\ell-1}) \in r^X$. By (A2), for each $n \in \mathbb{N}$, there is $(C_0^n, \dots, C_{\ell-1}^n) \in r^{\mathcal{C}_n}$ with $x_i \in C_i^n$, for $i < \ell$. Since χ_n is an epimorphism, there are $(u_0^n, \dots, u_{\ell-1}^n) \in r^{\mathbb{X}}$ such that $\chi_n(u_i^n) = C_i^n$, for $i < \ell$. Up to a subsequence $\big((u_0^n, \dots, u_{\ell-1}^n)\big)_{n\in\mathbb{N}}$ converges to $(u_0, \dots, u_{\ell-1})$, which belongs to $r^{\mathbb{X}}$, by closure. Since $x_i \in \chi_n(u_i^n)$, for each $n \in \mathbb{N}$, it follows that $q(u_i) = x_i$ for each $i < \ell$. Therefore q is an epimorphism.

It remains to show that q induces an isomorphism from $\mathbb{X}/R^{\mathbb{X}}$ to X. If $u, v \in \mathbb{X}$ then:

$$q(u) = q(v) \Leftrightarrow \forall n \in \mathbb{N}\ \chi_n(u) \cap \chi_n(v) \neq \emptyset \Leftrightarrow u\, R^{\mathbb{X}}\, v,$$

so we are done. $\square$

Definition 1.4.4. If $\mathcal{G}$ is a family of finite $\mathcal{L}_R$-structures, we denote by $[\![\mathcal{G}]\!]$ the class of compact metrizable $\mathcal{L}_R$-quotients which admit a $\mathcal{G}$-suitable sequence.

In general, $[\![\mathcal{G}]\!]$ is a subclass of the $\mathcal{L}_R$-quotients which are approximated by fine projective sequences from $\mathcal{G}$, but in most concrete situations the two classes coincide, see Theorem 3.4.3 or [IS06, BK15, PS18], for instance.

$\mathcal{G}$-suitable sequences create a bridge between the topological properties of $[\![\mathcal{G}]\!]$ and the combinatorial properties of $\mathcal{G}$. We exploit such bridge in Theorem 4.1.2, where we translate the combinatorial condition of projective amalgamation to obtain a topological characterization of the Fraïssé fence.

In many situations, the $\mathcal{L}_R$-quotients which admit $\mathcal{G}$-suitable sequences can also be understood by way of $\mathcal{G}$-like open covers. One such case is when language $\mathcal{L}_R$ is finite.

Lemma 1.4.5. *Let $\mathcal{G}$ be a family of finite structures in a finite language $\mathcal{L}_R$, and let $X \in [\![\mathcal{G}]\!]$. Then any open cover of X is refined by a $\mathcal{G}$-like open cover.*

Proof. Fix a compatible metric d on X. Let $\mathcal{U}$ be an open cover of X and let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a $\mathcal{G}$-suitable sequence of X. Let δ be the the Lebesgue covering number for $\mathcal{U}$. By Lemma 1.2.9 there is $n \in \mathbb{N}$ be such that $\mathcal{C}_n$ has mesh less than $\delta/2$. Say that $\mathcal{C}_n = \{C_a \mid a \in A\}$ is A-like, for some $A \in \mathcal{G}$. We show that we can enlarge the elements of $\mathcal{C}_n$ by a sufficiently small amount such that A-likeness is preserved.

For each $a \in A$, let $x_a \in C_a \setminus \bigcup_{a' \neq a} C_{a'}$ be given by (A0) for $\mathcal{C}_n$ and let $\delta_0 = \min\left\{ d\left(x_a, \bigcup_{a' \neq a} C_{a'}\right) \,\middle|\, a \in A \right\}$. Let $\delta_1 = \min\{d(C_a, C_{a'}) \mid a, a' \in A, C_a \cap C_{a'} = \emptyset\}$.

Fix $r \in \mathcal{L}_R \setminus \{R\}$ of arity ℓ. For each $\boldsymbol{a} = (a_0, \ldots, a_{\ell-1}) \in r^A$, fix $(x_0^{\boldsymbol{a}}, \ldots, x_{\ell-1}^{\boldsymbol{a}}) \in r^X$, such that $x_i^{\boldsymbol{a}} \in C_{a_i} \setminus \bigcup_{a \neq a_i} C_a$, for $i < \ell$, which is given by (A3) for $\mathcal{C}_n$. Let

$$\delta_{\boldsymbol{a}} = \min\left\{ d\left(x_i^{\boldsymbol{a}}, \bigcup_{a \neq a_i} C_a\right) \,\middle|\, i < \ell \right\},$$

and $\delta_r = \min\{\delta_{\boldsymbol{a}} \mid \boldsymbol{a} \in r^A\}$. Let $\delta_3 = \min\{\delta_r \mid r \in \mathcal{L}_R \setminus \{R\}\}$, which exists since $\mathcal{L}_R$ is finite.

Finally, let $\varepsilon < \min\{\delta/2, \delta_0, \delta_1, \delta_3\}$. For each $a \in A$, let $V_a = \{x \in X \mid d(x, C_a) < \varepsilon\}$, and let $\mathcal{V} = \{V_a \mid a \in A\}$. Then $\mathcal{V}$ refines $\mathcal{U}$, since its mesh is less than $\delta/2 + \varepsilon < \delta$. It also holds that $\mathcal{V}$ is A-like: (A0) holds since $\varepsilon < \delta_0$; (A1) holds because $\varepsilon < \delta_1$, so $a \, R^A \, a'$ if and only if $V_a \cap V_{a'} \neq \emptyset$.

Fix $r \in \mathcal{L}_R \setminus \{R\}$ of arity ℓ. Property (A2) for r is immediate since $C_a \subseteq V_a$ for each $a \in A$. So suppose $\boldsymbol{a} = (a_0, \ldots, a_{\ell-1}) \in r^A$. Then $(x_0^{\boldsymbol{a}}, \ldots, x_{\ell-1}^{\boldsymbol{a}}) \in r^X$ and $x_i^{\boldsymbol{a}} \in V_{a_i} \setminus \bigcup_{a \neq a_i} V_a$, since $\varepsilon < \delta_3 \leq \delta_r \leq \delta_{\boldsymbol{a}}$. Therefore (A3) holds. □

Approximate projective homogeneity

When $\mathbb{L}$ is the projective Fraïssé limit of a projective Fraïssé family $\mathcal{G}$ of finite $\mathcal{L}_R$-structures, it satisfies (L3) — projective ultrahomogeneity — with respect to $\mathcal{G}$. We establish conditions under which an approximate version of (L3) holds for $\mathbb{L}/R^{\mathbb{L}}$ with respect to $[\![\mathcal{G}]\!]$.

Let $\mathbb{L}$ be an $\mathcal{L}_R$-prespace and $\mathcal{G}$ a family of finite $\mathcal{L}_R$-structures. We consider the following property, reminiscent of (L2):

(SL2) For any $A \in \mathcal{G}$ and any A-like open cover $\mathcal{U} = \{U_a \mid a \in A\}$ of $\mathbb{L}/R^{\mathbb{L}}$ there is an epimorphism $\varphi : \mathbb{L} \to A$ such that $[\![a]\!]_\varphi \subseteq U_a$ for each $a \in A$.

Theorem 1.4.6 (Approximate projective homogeneity). *Let $\mathcal{L}_R$ be finite and $\mathcal{G}$ be a projective Fraïssé family of finite $\mathcal{L}_R$-structures with projective Fraïssé limit $\mathbb{L}$. Suppose that $\mathbb{L}$ is a prespace and that it satisfies (SL2). Then for any $X \in [\![\mathcal{G}]\!]$, epimorphisms $f_0, f_1 : \mathbb{L}/R^{\mathbb{L}} \to X$, and any open cover $\mathcal{V}$ of X, there is $\alpha \in \mathrm{Aut}(\mathbb{L})$ such that $f_0\alpha^*$ and f_1 are $\mathcal{V}$-close, that is, for each $x \in \mathbb{L}/R^{\mathbb{L}}$ there is $V \in \mathcal{V}$ such that $f_0\alpha^*(x), f_1(x) \in V$.*

Proof. Let $p : \mathbb{L} \to \mathbb{L}/R^{\mathbb{L}}$ denote the quotient map. By Lemma 1.4.5 there are $A \in \mathcal{G}$ and an A-like open cover $\mathcal{V}' = \{V_a \mid a \in A\}$ of X refining $\mathcal{V}$. Consider the open covers $f_i^{-1}\mathcal{V}' = \{f_i^{-1}(V_a) \mid a \in A\}$, for $i \leq 1$. By Lemma 1.2.7, these are A-like, so by (SL2) there are epimorphisms $\varphi_0, \varphi_1 : \mathbb{L} \to A$ such that $[\![a]\!]_{\varphi_i} \subseteq f_i^{-1}(V_a)$, for each $a \in A$ and $i \leq 1$.

By (L3) there is $\alpha \in \mathrm{Aut}(\mathbb{L})$ such that $\varphi_0\alpha = \varphi_1$. Fix $x \in \mathbb{L}/R^{\mathbb{L}}$ and $u \in p^{-1}(x)$. Then $f_0\alpha^*(x), f_1(x) \in V_{\varphi_1(u)}$. Indeed, $x \in [\![a]\!]_{\varphi_i} \subseteq f_i^{-1}(V_{\varphi_i(u)})$, for $i \leq 1$, so

$f_1(x) \in V_{\varphi_1}(u)$. On the other hand, $\alpha^*(x) = p\alpha(u) \in [\![\varphi_1(u)]\!]_{\varphi_0} \subseteq f_0^{-1}(V_{\varphi_1(u)})$, so $f_0\alpha^*(x) \in V_{\varphi_1(u)}$. $\square$

Corollary 1.4.7. *Let $\mathcal{L}_R$ be finite and $\mathcal{G}$ be a projective Fraïssé family of finite $\mathcal{L}_R$-structures with projective Fraïssé limit $\mathbb{L}$. Suppose that $\mathbb{L}$ is a prespace and that it satisfies (SL2). If $\mathbb{L}/R^{\mathbb{L}} \in [\![G]\!]$, then* $\mathrm{Aut}(\mathbb{L})$ *embeds densely in* $\mathrm{Aut}(\mathbb{L}/R^{\mathbb{L}})$.

Proof. Let $p : \mathbb{L} \to \mathbb{L}/R^{\mathbb{L}}$ denote the quotient map. By Proposition 1.4.3, p is irreducible and, by Remark 1.3.6, p^* is an embedding.

Fix a compatible metric d on $\mathbb{L}/R^{\mathbb{L}}$ and consider the corresponding supremum metric $d_{\sup}$ on $\mathrm{Aut}(\mathbb{L}/R^{\mathbb{L}})$. Let $h \in \mathrm{Aut}(\mathbb{L}/R^{\mathbb{L}})$ and $\varepsilon > 0$. Let $\mathcal{V}$ be an open cover of $\mathbb{L}/R^{\mathbb{L}}$ of mesh less than ε. We can thus apply Theorem 1.4.6 with $f_0 = \mathrm{id}_{\mathbb{L}/R^{\mathbb{L}}}, f_1 = h$ to find $\alpha \in \mathrm{Aut}(\mathbb{L})$ such that α^*, h are $\mathcal{V}$-close, that is, such that $d_{\sup}(\alpha^*, h) < \varepsilon$. $\square$

The condition that $\mathrm{Aut}(\mathbb{L})$ embeds densely in $\mathrm{Aut}(\mathbb{L}/R^{\mathbb{L}})$ is of great importance because it allows to pass some of the dynamical information of the prespace to the quotient. For example, if the universal minimal flow of $\mathrm{Aut}(\mathbb{L})$ is metrizable, so is $\mathrm{Aut}(\mathbb{L}/R^{\mathbb{L}})$ ([BK19, Theorem 5.3]), and if $\mathrm{Aut}(\mathbb{L})$ has a dense conjugacy class, so does $\mathrm{Aut}(\mathbb{L}/R^{\mathbb{L}})$ (see Corollary 4.5.5).

It is therefore natural to ask the following question.

Question 1.4.8. Which compact metrizable structures are quotients of projective Fraïssé limits such that the automorphisms of the prespace embed densely in those of the quotient?

An answer to such question was provided by Panagiotopoulos in [Pan16], albeit in a more expressive setting, which we explore in the next section. In Section 1.6 we instead show that in the framework adopted in this dissertation there are indeed limitations.

1.5 A second order digression

In this section we are interested in relational languages $(\mathcal{L}, \mathcal{L}^2)$ with two sorts: the first order sort $\mathcal{L}$, and the second order sort $\mathcal{L}^2$. Elements of $\mathcal{L}$ are first order relational symbols as in Section 1.1, whereas elements of $\mathcal{L}^2$ are second order relational symbols. Let $\mathrm{RC}(A)$ denote the algebra of regular closed sets of A.

A compact metrizable $(\mathcal{L}, \mathcal{L}^2)$-structure A is such that the restriction to $\mathcal{L}$ is a compact metrizable $\mathcal{L}$-structure and the interpretation S^A of second order relational symbol $S \in \mathcal{L}^2$ of arity n is a subset of $\mathrm{RC}(A)^n$.

If A, A' are compact metrizable $(\mathcal{L}, \mathcal{L}^2)$-structures, a map $\varphi : A' \to A$ is an epimorphism if it a continuous surjection such that $r^A = \varphi^{(n)}[r^A]$, for every n-ary relation symbol $r \in \mathcal{L}$, and such that $(\varphi^{-1}(C_0), \dots, \varphi^{-1}(C_{n-1})) \in S^{A'}$ whenever $(C_0, \dots, C_{n-1}) \in S^A$, for any n-ary second order relation symbol $S \in \mathcal{L}^2$.

Remark 1.5.1. The definition is meaningful since the preimages of regular closed sets are regular closed. Indeed, $\varphi : A' \to A$ is a continuous map between compact spaces, so in particular a closed map. Therefore preimage and closure commute, so the preimage of the closure of an open set is the closure of the preimage of an open set. Since a closed set is regular if and only if it is the closure of an open set, it follows that the preimage of a regular closed set is regular.

If (A_n, φ_n^m) is a projective sequence of $(\mathcal{L}, \mathcal{L}^2)$-structures and epimorphisms, its projective limit $\mathbb{A}$ is the $(\mathcal{L}, \mathcal{L}^2)$-structure whose restriction to $\mathcal{L}$ is the projective limit as defined in Section 1.1 and

$$S^{\mathbb{A}} = \bigcup_{n \in \mathbb{N}} \big\{ \big(\varphi_n^{-1}(C_0), \dots, \varphi_n^{-1}(C_{\ell-1})\big) \mid (C_0, \dots, C_{\ell-1}) \in S^{A_n} \big\},$$

for any ℓ-ary $S \in \mathcal{L}^2$.

If A is an $(\mathcal{L}_R, \mathcal{L}^2)$-prespace, and $p : A \to A/R^A$ is the quotient map, we can endow A/R^A with a $\mathcal{L}^2$ structure by letting $(C_0, \dots, C_{\ell-1}) \in S^{A/R^A}$ if and only if $\big(\varphi^{-1}(C_0), \dots, \varphi^{-1}(C_{\ell-1})\big) \in S^A$, for any ℓ-ary $S \in \mathcal{L}^2$. It follows that p is an $(\mathcal{L}_R, \mathcal{L}^2)$-epimorphism.

Remark 1.5.2. As has been noted before (see [BK19, Proposition 3.6]), projective Fraïssé theory of zero dimensional compact metrizable structures can be understood as the direct Fraïssé theory of Boolean algebras with additional structure, via Stone duality. Indeed, let $\mathrm{Clop}(A) \subseteq \mathrm{RC}(A)$ be the Boolean algebra of clopen subsets of A. To any n-ary first order relation r^A we can associate a relation $S_r^{\mathrm{Clop}(A)} \subseteq \mathrm{Clop}(A)^n$ by letting $(C_0, \dots, C_{n-1}) \in S_r^{\mathrm{Clop}(A)}$ if and only if there is $(x_0, \dots, x_{n-1}) \in r^A$, with $x_i \in C_i$, for $i < n$. Then φ is an epimorphism if and only if $C \mapsto \varphi^{-1}(C)$ is an embedding of Boolean algebras with these additional relations. For a finite structure A, it holds that $\mathrm{Clop}(A) = \mathrm{RC}(A)$, so the second order relations on A introduced above correspond exactly to first order relations on $\mathrm{Clop}(A)$, whereas first order relations on A give rise to exactly those relations on $\mathrm{Clop}(A)$ which are generated by their restriction to the atoms.

Some works in projective Fraïssé theory have considered classes of finite $\mathcal{L}$-structures with restricted epimorphisms, that is, where the class of relevant morphisms is a subclass of epimorphisms which contains the identities and is closed under composition.

This is the case in [PS18], in which the relevant morphisms are the connected epimorphisms — those such that the preimage of an R-connected subset is R-connected. It is clear, in this particular case, that the condition can be rephrased by adding a unary second order predicate: that of being an R-connected subset. We show that this is indeed the case more generally for all classes of restricted epimorphisms.

Lemma 1.5.3. *Let $\mathcal{G}$ be a family of finite $\mathcal{L}$-structures. Let Φ be a family of epimorphisms of $\mathcal{G}$-structures which is closed under composition and such that $\mathrm{id}_A \in \Phi$ for* *each $A \in \mathcal{G}$. Then there exist a second order language $\mathcal{L}^2$ and an expansion A^* of each $A \in \mathcal{G}$ to a $(\mathcal{L}, \mathcal{L}^2)$-structure, such that $\varphi : B \to A$ is an $(\mathcal{L}, \mathcal{L}^2)$-epimorphism from B^* to A^* if and only if $\varphi \in \Phi$.*

Proof. Let $\mathcal{L}^2 = \{S_A \mid A \in \mathcal{G}\}$, where S_A is a second order relation symbol of arity $|A|$. For each $A \in \mathcal{G}$, fix an enumeration $a_0, \dots, a_{|A|-1}$ of A.

For each $B \in \mathcal{G}$, let B^* be the $(\mathcal{L}, \mathcal{L}^2)$-structure whose restriction to $\mathcal{L}$ is B, and such that:

$$S_A^B = \big\{\big(\varphi^{-1}(a_0), \dots, \varphi^{-1}(a_{|A|-1})\big) \mid \varphi : B \to A,\ \varphi \in \Phi\big\},$$

for each $A \in \mathcal{G}$.

If $\varphi : B' \to B$, $\varphi \in \Phi$, and $\big(C_0, \dots, C_{|A|-1}\big) \in S_A^B$, then there is an epimorphism $\psi : B \to A$, $\psi \in \Phi$, such that $\big(C_0, \dots, C_{|A|-1}\big) = \big(\psi^{-1}(a_0), \dots, \psi^{-1}(a_{|A|-1})\big)$, so $\big(\varphi^{-1}(C_0), \dots, \varphi^{-1}(C_{|A|-1})\big) = \big((\psi\varphi)^{-1}(a_0), \dots, (\psi\varphi)^{-1}(a_{|A|-1})\big) \in S_A^{B'}$, since $\psi\varphi \in \Phi$. It follows that $\varphi : B'^* \to B^*$ is an $(\mathcal{L}, \mathcal{L}^2)$-epimorphism.

On the other hand, if $\varphi : B \to A$ is an $(\mathcal{L}, \mathcal{L}^2)$-epimorphism from B^* to A^*, then $\big(\varphi^{-1}(a_0), \dots, \varphi^{-1}(a_{|A|-1})\big) \in S_A^B$, so there exists $\psi \in \Phi$ such that $\big(\varphi^{-1}(a_0), \dots, \varphi^{-1}(a_{|A|-1})\big) = \big(\psi^{-1}(a_0), \dots, \psi^{-1}(a_{|A|-1})\big)$, that is, $\psi = \varphi$. □

This framework also encompasses that of *dual relations* presented in [Pan16]. An n-ary dual relation on A is a collection of clopen, ordered, n-partitions of A, that is, a subset of $\mathrm{Clop}(A)^n$ of elements $(C_0, \dots, C_{n-1})$ such that $\{C_0, \dots, C_{n-1}\}$ is a partition of A. These are a particular case of second order relations. Notice, for example, that S_A^B defined in the proof of Lemma 1.5.3 is a dual relation.

We can therefore state the main result of [Pan16] in terms of second order relations.

Theorem 1.5.4 ([Pan16, Theorem 5.2])**.** *Let X be a compact metrizable space and G a closed subgroup of $\mathrm{Homeo}(X)$. There is a two-sorted language $(\{R\}, \mathcal{L}^2)$, such that:*

- *X can be endowed with a $(\{R\}, \mathcal{L}^2)$-quotient structure with $\mathrm{cl}(\mathrm{Aut}(X)) = G$;*
- *There is a projective Fraïssé family of finite $(\{R\}, \mathcal{L}^2)$-structures, whose projective Fraïssé limit $\mathbb{L}$ is a close prespace of X;*
- *The quotient map $p : \mathbb{L} \to X$ induces an embedding $p^* : \mathrm{Aut}(\mathbb{L}) \to G$ with dense image.*

The above theorem therefore provides an answer to Question 1.4.8, in the context of second-order relations, or equivalently, restricted epimorphisms.

1.6 A negative result

In this section we show that a vast class of spaces, which includes closed manifolds of dimension greater than 1 and the Hilbert cube $[0, 1]^{\mathbb{N}}$, is not amenable to being studied via (first order) projective Fraïssé theory, in the sense of Question 1.4.8. In particular, it shows that Theorem 1.5.4 is not true in the context of first order languages.

Theorem 1.6.1. *Let X be an infinite ω-homogeneous compact metrizable space with an $\mathcal{L}_R$-quotient structure such that* $\mathrm{Aut}(X) = \mathrm{Homeo}(X)$. *Suppose there is a projective Fraïssé family $\mathcal{G}$ of finite $\mathcal{L}_R$-structures, whose projective Fraïssé limit $\mathbb{L}$ is a close prespace of X. If the quotient map $p : \mathbb{L} \to X$ induces an embedding $p^* : \mathrm{Aut}(\mathbb{L}) \to$* $\mathrm{Homeo}(X)$ *with dense image, then X is the Cantor space.*

Proof. Let (A_n, φ_n^m) be a fundamental sequence in $\mathcal{G}$. For any space Y and any $(x_0, \ldots, x_{\ell-1}) \in Y^\ell$, let:

$$[(x_0, \ldots, x_{\ell-1})]_{\mathrm{id}} = \left\{ (x'_0, \ldots, x'_{\ell-1}) \in Y^\ell \;\middle|\; \forall i, j < \ell \; x'_i = x'_j \text{ iff } x_i = x_j \right\}.$$

Fix $r \in \mathcal{L}_R \setminus \{R\}$ of arity ℓ.

Claim 1.6.1.1. If $(x_0, \ldots, x_{\ell-1}) \in r^X$ then $[(x_0, \ldots, x_{\ell-1})]_{\mathrm{id}} \subseteq r^X$.

Proof. For each $(x'_0, \ldots, x'_{\ell-1}) \in [(x_0, \ldots, x_{\ell-1})]_{\mathrm{id}}$, the bijection b mapping $x_i \mapsto x'_i$ for each $i < \ell$ is well defined. Since X is ℓ-homogeneous there is $g \in \mathrm{Homeo}(X)$ extending b. Fix a compatible metric d on X and let $d_{\sup}$ be the relative supremum metric on $\mathrm{Homeo}(X)$. For any $\varepsilon > 0$, there is $\alpha \in \mathrm{Aut}(\mathbb{L})$ such that $d_{\sup}(\alpha^*, g) < \varepsilon$. In particular $d(\alpha^*(x_i), x'_i) < \varepsilon$ for each $i < \ell$. But $(\alpha^*(x_0), \ldots, \alpha^*(x_{\ell-1})) \in r^X$ as $\alpha^* \in \mathrm{Aut}(X)$, and r^X is closed, so $(x'_0, \ldots, x'_{\ell-1}) \in r^X$. □

Suppose $u \in \mathbb{L}$ is an isolated point. Since p is irreducible, it is one-to-one on u, so also $p(u)$ is isolated. But X is infinite and 1-homogeneous, so it is perfect, a contradiction. So $\mathbb{L}$ is perfect.

Claim 1.6.1.2. If $(u_0, \ldots, u_{\ell-1}) \in r^{\mathbb{L}}$ then $[(u_0, \ldots, u_{\ell-1})]_{\mathrm{id}} \subseteq r^{\mathbb{L}}$.

Proof. Fix a compatible metric d on $\mathbb{L}$. Since p is irreducible, by Lemma 1.3.4, for each $\varepsilon > 0$, there exists $(v_0, \ldots, v_{\ell-1}) \in [(u_0, \ldots, u_{\ell-1})]_{\mathrm{id}}$ such that $d(u_i, v_i) < \varepsilon$ and p is one-to-one on v_i, for each $i < \ell$. Then $p^{(\ell)}(v_0, \ldots, v_{\ell-1}) \in \left[p^{(\ell)}(u_0, \ldots, u_{\ell-1})\right]_{\mathrm{id}}$ and $p^{(\ell)}(u_0, \ldots, u_{\ell-1}) \in r^X$, as p is an epimorphism. By Claim 1.6.1.1, $p^{(\ell)}(v_0, \ldots, v_{\ell-1}) \in r^X$, but since p is one-to-one on v_i, for $i < \ell$, it follows that $(v_0, \ldots, v_{\ell-1}) \in r^{\mathbb{L}}$. We conclude by closure of $r^{\mathbb{L}}$. □

Claim 1.6.1.3. For each $n \in \mathbb{N}$, let $A'_n = A_n{\restriction_{\{R\}}}$ be the reduct of A_n with respect to the language $\{R\} \subseteq \mathcal{L}_R$. For any $m \geq n$ and any $\{R\}$-epimorphism $\psi : A'_m \to A'_n$, it holds that $\psi : A_m \to A_n$ is an $\mathcal{L}_R$-epimorphism.

Proof. Let $r \in \mathcal{L}_R \setminus \{R\}$ be of arity ℓ. Suppose that $(a_0, \ldots, a_{\ell-1}) \in r^{A_m}$ and let $(u_0, \ldots, u_{\ell-1}) \in r^{\mathbb{L}}$ be such that $\varphi_m(u_i) = a_i$. Since $\mathbb{L}$ is perfect and $\varphi_n^{-1}(b)$ is clopen for each $b \in A_n$, there is a tuple $(v_0, \ldots, v_{\ell-1}) \in [(u_0, \ldots, u_{\ell-1})]_{\mathrm{id}}$ such that $v_i \in \varphi_n^{-1}(\psi(a_i))$ for each $i < \ell$. It follows that $(v_0, \ldots, v_{\ell-1}) \in r^{\mathbb{L}}$, so $(\psi(a_0), \ldots, \psi(a_{\ell-1})) \in r^{A_n}$.

Conversely, if $(b_0, \ldots, b_{\ell-1}) \in r^{A_n}$, let $(v_0, \ldots, v_{\ell-1}) \in r^{\mathbb{L}}$ be such that $\varphi_n(v_i) = b_i$. Since $\mathbb{L}$ is perfect and $\varphi_m^{-1}(a)$ is clopen, for each $a \in A_m$, there is a tuple $(u_0, \ldots, u_{\ell-1}) \in [(v_0, \ldots, v_{\ell-1})]_{\mathrm{id}}$ such that $u_i \in \varphi_n^{-1}(\psi^{-1}(b_i))$ for each $i < \ell$. It follows that $(u_0, \ldots, u_{\ell-1}) \in r^{\mathbb{L}}$ and therefore that $(\varphi_n(u_0), \ldots, \varphi_n(u_{\ell-1})) \in r^{A_n}$. But $\psi\varphi_n(u_i) = b_i$, so we are done. $\square$

Let $\mathbb{L}'$ be the reduct of $\mathbb{L}$ in the language $\{R\}$, or equivalently the projective limit of (A'_n, φ_n^m). Clearly any $\mathcal{L}_R$-epimorphism is an $\{R\}$-epimorphism, so each φ_n is. We prove that $\mathbb{L}'$ is a projective Fraïssé limit of a family of $\{R\}$-structures, namely $\{A'_n \mid n \in \mathbb{N}\}$, by showing that (A'_n, φ_n^m) is a fundamental sequence. Property (F1) is clear. Let $\psi : A'_n \to A'_k, \chi : A'_\ell \to A'_k$ be a $\{R\}$-epimorphisms. By Claim 1.6.1.3, ψ, χ are $\mathcal{L}_R$-epimorphisms $A_n \to A_k, A_\ell \to A_k$. By (F2) for (A_n, φ_n), there are $m \geq n$ and $\psi' : A_m \to A_\ell$ such that $\psi\varphi_n^m = \chi\psi'$. Since in particular ψ' is an $\{R\}$-epimorphism $A'_m \to A'_\ell$, we conclude.

It follows that $\mathbb{L}'$ is a projective Fraïssé limit and a close prespace of X. By [Cam10, Theorem 15], the only perfect compact metrizable $\{R\}$-quotients are:

- the Cantor space,
- a disjoint union of finitely many pseudo-arcs,
- a disjoint union of finitely many spaces $X = P \sqcup \bigcup_{i \in \mathbb{N}} Q_i$, where P is a pseudo-arc, each Q_i is a Cantor space clopen in X and $\bigcup_{i \in \mathbb{N}} Q_i$ is dense in X.

We conclude by noticing that the pseudo-arc is not 2-homogeneous (see [Usp00]), so the only ω-homogeneous space among the ones above is the Cantor space. $\square$

Question 1.6.2. Can the condition that $\mathbb{L}$ be a *close* prespace of X be dropped in the above theorem?

It is worth noting that all examples of quotients of projective Fraïssé limits in the literature which satisfy the requirements of Question 1.4.8 are, to the author's knowledge, one dimensional. It is unclear if the higher dimension is an intrinsic obstruction.

Chapter 2

Finitely representable spaces

In this chapter we explore the question of which compact metrizable spaces appear as domains of quotients of projective Fraïssé prespaces.

Definition 2.0.1. A compact metrizable space X is $\mathcal{L}_R$-*representable* if there exists a projective Fraïssé family of finite $\mathcal{L}_R$-structures with projective Fraïssé limit $\mathbb{L}$, such that $\mathbb{L}$ is a prespace and (the domain of) $\mathbb{L}/R^{\mathbb{L}}$ is homeomorphic to X.

A space X is *finitely representable* if it is $\mathcal{L}_R$-representable for some finite $\mathcal{L}_R$.

In this terminology, when $\mathcal{L}_R = \{R\}$, the $\mathcal{L}_R$-representable spaces have been characterised in [Cam10].

2.1 Some preliminary facts

Proposition 2.1.1. *Suppose $\mathcal{G}$ is a projective Fraïssé family in the language $\mathcal{L}_R$. Let $\mathbb{L}$ be a projective Fraïssé limit of $\mathcal{G}$.*

1. *If R is interpreted by all structures in $\mathcal{G}$ as a reflexive relation, then $R^{\mathbb{L}}$ is reflexive as well.*

2. *If R is interpreted by all structures in $\mathcal{G}$ as a symmetric relation, then $R^{\mathbb{L}}$ is symmetric as well.*

3. *If R is interpreted by all structures in $\mathcal{G}$ as an anti-symmetric relation, then $R^{\mathbb{L}}$ is anti-symmetric as well.*

4. *If R is interpreted by all structures in $\mathcal{G}$ as a transitive relation, then $R^{\mathbb{L}}$ is transitive as well.*

5. *If R is interpreted by all structures in $\mathcal{G}$ as a total relation, then $R^{\mathbb{L}}$ is total as well.*

6. *If R is interpreted by all structures in $\mathcal{G}$ as having a first (respectively, last) element, then $R^{\mathbb{L}}$ has a first (respectively, last) element as well.*

7. *If R is interpreted by all structures in $\mathcal{G}$ as a connected relation, then for any partition $\{U, V\}$ of $\mathbb{L}$ into clopen sets there are $x \in U, y \in V$ with $xR^{\mathbb{L}}y$.*

Proof. In this proof we will use property (L2) extensively.

(1) and (2) The proof is similar to the argument carried out in the proof of [IS06, Lemma 4.1].

(3) Let $x, y \in \mathbb{L}$ be distinct elements such that $x\, R^{\mathbb{L}}\, y\, R^{\mathbb{L}}\, x$. Pick a clopen subset U of $\mathbb{L}$ such that $x \in U, y \notin U$ and find $A \in \mathcal{G}$ with an epimorphism $\varphi : \mathbb{L} \to A$ refining $\{U, \mathbb{L} \setminus U\}$. Then $\varphi(x), \varphi(y)$ are distinct and $\varphi(x)\; R^A\; \varphi(y)\; R^A\; \varphi(x)$.

(4) Let $x, y, z \in \mathbb{L}$, with $x\; R^{\mathbb{L}}\; y\; R^{\mathbb{L}}\; z$. Since $R^{\mathbb{L}}$ is closed, it is enough to show that for any neighbourhoods U of x and V of z there are $x' \in U$, $z' \in V$ with $x'\; R^{\mathbb{L}}\; z'$. Let $U' \subseteq U, V' \subseteq V$ be clopen neighbourhoods of x, z, respectively, with $U' = V'$ if $x = z$ and $U' \cap V' = \emptyset$ otherwise. Let $A \in \mathcal{G}$ and $\varphi : \mathbb{L} \to A$ be an epimorphism refining the clopen covering $\{U', V', \mathbb{L} \setminus (U' \cup V')\}$. Since $\varphi(x)\; R^A\; \varphi(z)$, there are $x' \in U'$, $z' \in V'$ with $\varphi(x) = \varphi(x')$, $\varphi(z) = \varphi(z')$, $x'\; R^{\mathbb{L}}\; z'$.

(5) It is enough to show that, given $x, y \in \mathbb{L}$, whenever U, V are clopen neighbourhoods of x, y respectively, there are $x' \in U$, $y' \in V$ such that either $x'\, R^{\mathbb{L}}\, y'$ or $y'\, R^{\mathbb{L}}\, x'$. Moreover, if $x = y$ it can be assumed that $U = V$, while for $x \neq y$ one can take $U \cap V = \emptyset$. Let $A \in \mathcal{G}$ with an epimorphism $\varphi : \mathbb{L} \to A$ refining the clopen covering $\{U, V, \mathbb{L} \setminus (U \cup V)\}$. Since $\varphi(x)\, R^A\, \varphi(y)$ or $\varphi(y)\, R^A\, \varphi(x)$, there are $x' \in U$, $y' \in V$ such that $\varphi(x') = \varphi(x)$, $\varphi(y') = \varphi(y)$ and either $x'\; R^{\mathbb{L}}\; y'$ or $y'\; R^{\mathbb{L}}\; x'$.

(6) Argue for the first element, the situation for the last being similar. Fix a compatible complete metric on $\mathbb{L}$ and, for each positive integer n, let $\mathcal{U}_n$ be a partition of $\mathbb{L}$ with clopen sets of diameter less than $1/n$ such that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$. Let $\varphi_n : \mathbb{L} \to A_n$ be an epimorphism refining $\mathcal{U}_n$ onto some $A_n \in \mathcal{G}$. Let $x_n \in \mathbb{L}$ be such that $\varphi_n(x_n)$ is the first element of R^{A_n} and fix a limit point x of the sequence x_n, in order to show that $\forall y \in \mathbb{L}\; x\; R^{\mathbb{L}}\; y$. For this it is enough to prove that given clopen neighbourhoods U, V of x, y, respectively, where it can be assumed that $U = V$ if $x = y$ and that $U \cap V = \emptyset$ if $x \neq y$, there are $x' \in U$, $y' \in V$ with $x'\; R^{\mathbb{L}}\; y'$. Take n such that if $x \in W \in \mathcal{U}_n$ and $y \in W' \in \mathcal{U}_n$, then $W \subseteq U, W' \subseteq V$. Let $n' \geq n$ be such that $x_{n'} \in W$. Notice that $\varphi_{n'}$ refines $\{W, W', \mathbb{L} \setminus (W \cup W')\}$. Since $\varphi_{n'}(x_{n'})\, R^{A_{n'}}\, \varphi(y)$, there are $x' \in W$, $y' \in W'$ such that $\varphi_{n'}(x') = \varphi_{n'}(x_{n'})$, $\varphi_{n'}(y') = \varphi_{n'}(y)$, $x'\; R^{\mathbb{L}}\; y'$.

(7) As for the argument in the proof of [IS06, Lemma 4.3]. □

Notice that for (1), (2), (5), (6) the converse holds as well.

Now we show that if one admits infinite languages, then every compact metrizable space is homeomorphic to the quotient of a projective Fraïssé limit. Consequently, in the sequel of this chapter we will be interested in studying what kind of spaces can be obtained with finite languages, with the hope that this notion might provide an interesting topological dividing line.

Lemma 2.1.2. *Let $\mathcal{L}$ be any language and let $\{A_n \mid n \in \mathbb{N}\}$ be a family of finite $\mathcal{L}$-structures. If for every $n \leq m$ there is exactly one epimorphism $\varphi_n^m : A_m \to A_n$, then (A_n, φ_n^m) is a fundamental sequence.*

Proof. First notice that from the hypotheses it follows that for any n, m there is at most one epimorphism $A_m \to A_n$. If $n \leq m$ this is in the hypotheses; on the other hand, if $n > m$, the existence of an epimorphism $\psi : A_m \to A_n$ implies that ψ and φ_m^n are actually isomorphisms; if there were two different isomorphisms $A_m \to A_n$, their compositions with φ_m^n would yield two different isomorphisms $A_n \to A_n$.

Consequently, given any two epimorphisms $\psi_1 : A_h \to A_k$, $\psi_2 : A_p \to A_k$ and letting $m = \max\{h, p\}$, one has $\psi_1 \varphi_h^m = \psi_2 \varphi_p^m$. $\square$

Proposition 2.1.3. *Let $\mathcal{L}_R = \{R, \rho_s\}_{s \in 2^{<\omega}}$, where the ρ_s are unary relation symbols for all $s \in 2^{<\omega}$. Then every compact metrizable space is $\mathcal{L}_R$-representable.*

Proof. Let X be a compact metrizable space and let $\equiv$ be a closed equivalence relation on $2^{\mathbb{N}}$ such that $X \backsimeq 2^{\mathbb{N}}/\equiv$. Define $\mathcal{L}_R$-structures $A_n = (2^n, R^{A_n}, \rho_s^{A_n})_{s \in 2^{<\omega}}$ by letting

$$\begin{aligned} u\, R^{A_n}\, u' \quad &\Leftrightarrow \quad \exists x, x' \in 2^{\mathbb{N}}\ (u \subseteq x \wedge u' \subseteq x' \wedge x \equiv x') \\ \rho_s^{A_n}(u) \quad &\Leftrightarrow \quad s \subseteq u \vee u \subseteq s \end{aligned}$$

Now notice that, given $n \leq m$, the only epimorphism $A_m \to A_n$ is the restriction map φ_n^m defined by $\varphi_n^m(w) = w{\restriction_n}$. Indeed, if $w, w' \in 2^m$ are such that $w\, R^{A_m}\, w'$, let $x, x' \in 2^{\mathbb{N}}$ with $w \subseteq x$, $w' \subseteq x'$, $x \equiv x'$; since $x{\restriction_n} = \varphi_n^m(w)$, $x'{\restriction_n} = \varphi_n^m(w')$, it follows that $\varphi_n^m(w)\, R^{A_n}\, \varphi_n^m(w')$. Moreover, if $w \in 2^m$ satisfies $\rho_s^{A_m}(w)$ for some $s \in 2^{<\omega}$, so that w is compatible with s, its restriction $w{\restriction_n}$ is compatible with s as well, so $\rho_s^{A_n}(\varphi_n^m(w))$ holds. Conversely, assume first that $u, u' \in 2^n$ fulfil $u\, R^{A_n}\, u'$ and let $x, x' \in 2^{\mathbb{N}}$ such that $u \subseteq x$, $u' \subseteq x'$, $x \equiv x'$; then $x{\restriction_m}\, R^{A_m}\, x'{\restriction_m}$, $\varphi_n^m(x{\restriction_m}) = u$, $\varphi_n^m(x'{\restriction_m}) = u'$. Finally, suppose that $s \in 2^{<\omega}, u \in 2^n$ are such that $\rho_s^{A_n}(u)$; then there is at least an element $w \in 2^m$ such that $\varphi_n^m(w) = w{\restriction_n} = u$ and w is compatible with s, so that $\rho_s^{A_m}(w)$. To see that φ_n^m is the unique epimorphism $A_m \to A_n$, notice that for any $w \in 2^m$, the unique element $u \in 2^n$ such that $\rho_w^{A_n}(u)$ is $w{\restriction_n}$.

Consequently, by Lemma 2.1.2, (A_n, φ_n^m) is a fundamental sequence. Let $\mathbb{L} = (2^{\mathbb{N}}, R^{\mathbb{L}}, \rho_s^{\mathbb{L}})_{s \in 2^{<\omega}}$ be its projective limit. It is now enough to prove $R^{\mathbb{L}} = \equiv$, so let $x, x' \in 2^{\mathbb{N}}$. If $x \equiv x'$, then $\forall n \in \mathbb{N}\ x{\restriction_n}\, R^{A_n}\, x'{\restriction_n}$, so that $x\, R^{\mathbb{L}}\, x'$. Conversely, if $x\, R^{\mathbb{L}}\, x'$, so that $\forall n \in \mathbb{N}\ x{\restriction_n}\, R^{A_n}\, x'{\restriction_n}$, for every $n \in \mathbb{N}$ there are $x_n, x'_n \in 2^{\mathbb{N}}$ such that $x{\restriction_n} \subseteq x_n$, $x'{\restriction_n} \subseteq x'_n$, $x_n \equiv x'_n$, so that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} x'_n = x'$, $x \equiv x'$, since $\equiv$ is closed. $\square$

Remark 2.1.4. In the above proof, the $\mathcal{L}_R$-structure on X is such that $\mathrm{Aut}(X) = \{\mathrm{id}\}$. Therefore it does not contradict Theorem 1.6.1.

2.2 Closure under topological operations

This section collects some closure properties of finitely representable spaces. We will need the following notion.

Definition 2.2.1. Let $\mathbb{L}$ be an $\mathcal{L}_R$-prespace. A point $x \in \mathbb{L}$ is *almost stable* if, for all $\alpha \in \mathrm{Aut}(\mathbb{L})$, one has that $\alpha(x)\, R^{\mathbb{L}}\, x$.

Notice that the set of almost stable points is invariant under the equivalence relation $R^{\mathbb{L}}$.

Theorem 2.2.2. *The finite disjoint sum of finitely representable spaces is finitely representable.*

Proof. It is enough to prove the result for the disjoint sum of two spaces[1]. So, for $i \in \{1,2\}$ let X_i be $\mathcal{L}_R^i$-representable for some finite $\mathcal{L}_R^i$, as witnessed by a projective Fraïssé family $\mathcal{G}_i$ with limit $\mathbb{L}_i$. We can sssume that $\mathcal{L}_R^1 \cap \mathcal{L}_R^2 = \{R\}$. Let $\mathcal{L}_R = \mathcal{L}_R^1 \cup \mathcal{L}_R^2 \cup \{P_1, P_2\}$, where P_1, P_2 are new unary relation symbols.

Given compact metrizable $\mathcal{L}_R^i$-structures A_i, for $i \in \{1,2\}$, define an $\mathcal{L}_R$-structure $A = A_1 \oplus A_2$ as follows:

- A is a disjoint union $A_1 \cup A_2$, with each A_i clopen in A;
- $R^A = R^{A_1} \cup R^{A_2}$;
- $P_i^A = A_i$;
- if $S \in \mathcal{L}_R^i$ is a relation symbol different from R, then $S^A = S^{A_i}$.

Notice that if $\psi_i : A_i \to B_i$ are $\mathcal{L}_R^i$-epimorphisms for $i \in \{1,2\}$, then $\psi_1 \cup \psi_2 : A_1 \oplus A_2 \to B_1 \oplus B_2$ is an $\mathcal{L}_R$-epimorphism. Conversely, if $\psi : A_1 \oplus A_2 \to B_1 \oplus B_2$ is an $\mathcal{L}_R$-epimorphism, then by the interpretations of symbols P_1, P_2, the restriction ψ_i of ψ to A_i has range included in — in fact, equal to — B_i; moreover $\psi_i : A_i \to B_i$ is an $\mathcal{L}_R^i$-epimorphism.

Define $\mathcal{G}$ as the class of $\mathcal{L}_R$-structures $A = (A, R^A, \ldots, P_1^A, P_2^A)$ of the form $A = A_1 \oplus A_2$, where $A_i \in \mathcal{G}_i$.

Claim 2.2.2.1. $\mathcal{G}$ is a projective Fraïssé family.

Proof of claim. (JPP): Let $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2 \in \mathcal{G}$. By (JPP) of $\mathcal{G}_i$, let $C_i \in \mathcal{G}_i$, with epimorphisms $\psi_i : C_i \to A_i$, $\theta_i : C_i \to B_i$. Set $C = C_1 \oplus C_2 \in \mathcal{G}$, $\psi = \psi_1 \cup \psi_2 : C \to A$, $\theta = \theta_1 \cup \theta_2 : C \to B$. Then ψ, θ are epimorphisms.

(AP): Let $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, $C = C_1 \oplus C_2 \in \mathcal{G}$, with epimorphisms $\psi : B \to A$, $\theta : C \to A$. So let $\psi_i = \psi_{\restriction B_i}$, $\theta_i = \theta_{\restriction C_i}$, then $\psi_i : B_i \to A_i$, $\theta_i : C_i \to A_i$

[1] Notice that for the sum of n spaces, a direct proof would provide a smaller language than the one resulting by iterating the construction in the proof.

are epimorphisms. By (AP) for $\mathcal{G}_i$, let $D_i \in \mathcal{G}_i$, $\psi'_i : D_i \to B_i$, $\theta'_i : D_i \to C_i$ be epimorphisms such that $\psi_i\psi'_i = \theta_i\theta'_i$. Let $D = D_1 \oplus D_2 \in \mathcal{G}$. So $\psi' = \psi'_1 \cup \psi'_2 : D \to B$, $\theta' = \theta'_1 \cup \theta'_2 : D \to C$ are epimorphisms such that $\psi\psi' = \theta\theta'$. $\square$

Let $\mathbb{L} = \mathbb{L}_1 \oplus \mathbb{L}_2$.

Claim 2.2.2.2. $\mathbb{L}$ is the projective Fraïssé limit of $\mathcal{G}$.

Proof of claim. It is enough to carry out the following three verifications:

- (L1) Let $A = A_1 \oplus A_2 \in \mathcal{G}$. By projective universality of $\mathbb{L}_i$, let $\psi_i : \mathbb{L}_i \to A_i$ be an epimorphism. Then $\psi = \psi_1 \cup \psi_2 : \mathbb{L} \to A$ is an epimorphism.
- (L2) Let $\mathcal{U}$ be a partition of $\mathbb{L}$ into clopen sets, which can be assumed to refine $\{\mathbb{L}_1, \mathbb{L}_2\}$. So $\mathcal{U} \cap \mathcal{P}(\mathbb{L}_i)$ is a partition of $\mathbb{L}_i$ into clopen sets. Let $A_i \in \mathcal{G}_i$ with an epimorphism $\psi_i : \mathbb{L}_i \to D_i$ refining $\mathcal{U} \cap \mathcal{P}(\mathbb{L}_i)$. So $\psi = \psi_1 \cup \psi_2 : \mathbb{L} \to A_1 \oplus A_2$ is an epimorphism refining $\mathcal{U}$.
- (L3) Let $A = A_1 \oplus A_2 \in \mathcal{G}$, with epimorphisms $\psi_1, \psi_2 : \mathbb{L} \to A$. So $\psi_{j \restriction \mathbb{L}_i}$ are epimorphisms $\mathbb{L}_i \to A_i$. By projective ultrahomogeneity of $\mathbb{L}_i$, let $\theta_i : \mathbb{L}_i \to \mathbb{L}_i$ be an isomorphism such that $\psi_{1 \restriction \mathbb{L}_i}\theta_i = \psi_{2 \restriction \mathbb{L}_i}$. So $\theta = \theta_1 \cup \theta_2 : \mathbb{L} \to \mathbb{L}$ is an isomorphism such that $\psi_1\theta = \psi_2$.

$\square$

Notice that $R^{\mathbb{L}}$ is an equivalence relation on $\mathbb{L}$ and $\mathbb{L}/R^{\mathbb{L}}$ is a disjoint sum of $\mathbb{L}_1/R^{\mathbb{L}_1}, \mathbb{L}_2/R^{\mathbb{L}_2}$, completing the proof. $\square$

For later use we remark that in the proof of Theorem 2.2.2, if x is an almost stable point in one of the $\mathbb{L}_i$, then x is almost stable also in the resulting $\mathbb{L}$.

Theorem 2.2.3. *The finite product of finitely representable spaces is finitely representable.*

Proof. It is enough to prove the assertion for products of two factors[2]. So, for $i \in \{1, 2\}$ let X_i be $\mathcal{L}^i_R$-representable, for some finite $\mathcal{L}^i_R$, as witnessed by a projective Fraïssé family $\mathcal{G}_i$ with limit $\mathbb{L}_i$. We can assume that $\mathcal{L}^1_R \cap \mathcal{L}^2_R = \{R\}$. Let $\mathcal{L}_R = \mathcal{L}^1_R \cup \mathcal{L}^2_R \cup \{r_1, r_2\}$, where r_1, r_2 are two new binary relation symbols. Let $\mathcal{G} = \{A \times B \mid A \in \mathcal{G}_1, B \in \mathcal{G}_2\}$ where:

- $(a, b)\, R^{A\times B}\, (a', b') \Leftrightarrow a\, R^A\, a' \wedge b\, R^B\, b'$;
- $S^{A\times B}((a_1, b_1), \ldots, (a_m, b_m)) \Leftrightarrow S^A(a_1, \ldots, a_m)$ for any m-ary relation symbol $S \in \mathcal{L}^1_R \setminus \{R\}$;

[2]Remarks about the language similar to those in Theorem 2.2.2 apply here.

- $S^{A\times B}((a_1,b_1),\dots,(a_m,b_m)) \Leftrightarrow S^B(b_1,\dots,b_m)$ for any m-ary relation symbol $S \in \mathcal{L}^2_R \setminus \{R\}$;
- $r_1^{A\times B}((a_1,b_1),(a_2,b_2)) \Leftrightarrow a_1 = a_2$;
- $r_2^{A\times B}((a_1,b_1),(a_2,b_2)) \Leftrightarrow b_1 = b_2$.

Claim 2.2.3.1. $\psi : A\times B \to C\times D$ is an epimorphism if and only if $\psi = \theta\times\gamma$ for some epimorphisms $\theta : A \to C, \gamma : B \to D$.

Proof of claim. Let $\psi : A\times B \to C\times D$ be an epimorphism. Since $r_1^{A\times B}((a,b_1),(a,b_2))$, from $r_1^{C\times D}(\psi(a,b_1),\psi(a,b_2))$ it follows that $\psi(a,b_1),\psi(a,b_2)$ have the same first component; similarly for $\psi(a_1,b),\psi(a_2,b)$. This means that $\psi = \theta\times\gamma$ for some surjective $\theta : A \to C$, $\gamma : B \to D$. It remains to prove that θ, and similarly γ, are epimorphisms.

Suppose $c\,R^C\,c'$. Since R^D is reflexive, by reflexivity of $R^{\mathbb{L}}$ and Proposition 2.1.1(1), it follows that for any $d \in D$ one has $(c,d)\ R^{C\times D}\ (c',d)$. So there are $(a,b),(a',b') \in A\times B$ such that $\psi(a,b) = (c,d)$, $\psi(a',b') = (c',d)$, $(a,b)\ R^{A\times B}\ (a',b')$. Consequently, $\theta(a) = c$, $\theta(a') = c'$, $a\ R^A\ a'$. Conversely, if $a\ R^A\ a'$, for any $b \in B$ one has $(a,b)\ R^{A\times B}$ (a',b), whence $(\theta(a),\gamma(b))\ R^{C\times D}\ (\theta(a'),\gamma(b))$, so $\theta(a)\ R^C\ \theta(a')$.

Let $S \in \mathcal{L}^1_R\setminus\{R\}$ be an m-ary relation symbol. If $S^C(c_1,\dots,c_m)$, for any $d \in D$ one has $S^{C\times D}((c_1,d),\dots,(c_m,d))$. Let $a_1,\dots,a_m \in A$, $b_1,\dots,b_m \in B$ with $\psi(a_1,b_1) = (c_1,d),\dots,\psi(a_m,b_m) = (c_m,d)$, $S^{A\times B}((a_1,b_1),\dots,(a_m,b_m))$. This implies $\theta(a_1) = c_1$, $\dots,\theta(a_m) = c_m$, $S^A(a_1,\dots,a_m)$. Conversely, whenever $S^A(a_1,\dots,a_m)$, picking any $b \in B$, one has $S^{A\times B}((a_1,b),\dots,(a_m,b))$, whence $S^{C\times D}(\psi(a_1,b),\dots,\psi(a_m,b))$, which allows to conclude that $S^C(\theta(a_1),\dots,\theta(a_m))$.

Assume now $\theta : A \to C$, $\gamma : B \to D$ are epimorphisms, and set $\psi = \theta\times\gamma$. Then, for any $(c,d),(c',d') \in C\times D$,

$$
\begin{aligned}
&(c,d)\ R^{C\times D}\ (c',d') \Leftrightarrow c\ R^C\ c' \wedge d\ R^D\ d' \Leftrightarrow \\
&\qquad \Leftrightarrow \exists a,a' \in A\ \exists b,b' \in B \\
&\qquad\quad (\theta(a) = c \wedge \theta(a') = c' \wedge \gamma(b) = d \wedge \gamma(b') = d' \wedge a\ R^A\ a' \wedge b\ R^B\ b') \Leftrightarrow \\
&\qquad \Leftrightarrow \exists a,a' \in A\ \exists b,b' \in B \\
&\qquad\quad (\psi(a,b) = (c,d) \wedge \psi(a',b') = (c',d') \wedge (a,b)\ R^{A\times B}\ (a',b')).
\end{aligned}
$$

If $S \in \mathcal{L}^1_R \setminus \{R\}$ is an m-ary relation symbol and $S^{A\times B}((a_1,b_1),\dots,(a_m,b_m))$, then $S^A(a_1,\dots,a_m)$, whence $S^C(\theta(a_1),\dots,\theta(a_m))$ and finally:

$$S^{C\times D}(\psi(a_1,b_1),\dots,\psi(a_m,b_m)).$$

Conversely, suppose $S^{C\times D}((c_1,d_1),\dots,(c_m,d_m))$, that is, $S^C(c_1,\dots,c_m)$. So there are $a_1,\dots,a_m \in A$ such that $\theta(a_1) = c_1,\dots,\theta(a_m) = c_m$, $S^A(a_1,\dots,a_m)$. Taking any $b_1,\dots,b_m \in B$ such that $\gamma(b_1) = d_1,\dots,\gamma(b_m) = d_m$, one has $\psi(a_1,b_1) = (c_1,d_1),\dots,\psi(a_m,b_m) = (c_m,d_m)$, $S^{A\times B}((a_1,b_1),\dots,(a_m,b_m))$. Similarly for symbols in $\mathcal{L}^2_R$. $\square$

Claim 2.2.3.2. $\mathcal{G}$ is a projective Fraïssé family.

Proof of claim. (JPP): Let $A \times B$, $C \times D \in \mathcal{G}$. By (JPP) of $\mathcal{G}_1$ and $\mathcal{G}_2$, let $E \in \mathcal{G}_1$, $F \in \mathcal{G}_2$ with epimorphisms $\psi_1 : E \to A$, $\psi_2 : E \to C$, $\theta_1 : F \to B$, $\theta_2 : F \to D$. Then $\psi_1 \times \theta_1 : E \times F \to A \times B$, $\psi_2 \times \theta_2 : E \times F \to C \times D$ are epimorphisms.

(AP): Let $A_1 \times A_2$, $B_1 \times B_2$, $C_1 \times C_2 \in \mathcal{G}$ with epimorphisms $\psi : B_1 \times B_2 \to A_1 \times A_2$, $\theta : C_1 \times C_2 \to A_1 \times A_2$. By the preceding claim, there are epimorphisms $\psi_1, \psi_2, \theta_1, \theta_2$ such that $\psi = \psi_1 \times \psi_2$, $\theta = \theta_1 \times \theta_2$. Using (AP) of $\mathcal{G}_1$, $\mathcal{G}_2$, let $D_1 \in \mathcal{G}_1$, $D_2 \in \mathcal{G}_2$ with epimorphisms $\gamma_1 : D_1 \to B_1$, $\rho_1 : D_1 \to C_1, \gamma_2 : D_2 \to B_2$, $\rho_2 : D_2 \to C_2$ be such that $\psi_1\gamma_1 = \theta_1\rho_1$, $\psi_2\gamma_2 = \theta_2\rho_2$. Thus $\gamma_1 \times \gamma_2 : D_1 \times D_2 \to B_1 \times B_2$, $\rho_1 \times \rho_2 : D_1 \times D_2 \to C_1 \times C_2$ are epimorphisms such that $\psi(\gamma_1 \times \gamma_2) = \theta(\rho_1 \times \rho_2)$. $\square$

Let now $(A_n, \varphi_n^m), (B_n, \rho_n^m)$ be fundamental sequences for $\mathcal{G}_1, \mathcal{G}_2$, respectively.

Claim 2.2.3.3. $(A_n \times B_n, \varphi_n^m \times \rho_n^m)$ is a fundamental sequence for $\mathcal{G}$.

Proof of claim. Let $A \times B \in \mathcal{G}$. There are $n, m \in \mathbb{N}$ and epimorphisms $\psi : A_n \to A$, $\theta : B_m \to B$. If $n \leq m$, then $(\psi\varphi_n^m) \times \theta : A_m \times B_m \to A \times B$ is an epimorphism; otherwise, $\psi \times (\theta\rho_m^n) : A_n \times B_n \to A \times B$ is.

Let now $E_1 \times E_2$, $F_1 \times F_2 \in \mathcal{G}$, $n \in \mathbb{N}$, with epimorphisms $\psi_1 \times \psi_2 : F_1 \times F_2 \to E_1 \times E_2$, $\theta_1 \times \theta_2 : A_n \times B_n \to E_1 \times E_2$. Let $m, m' \geq n$ with epimorphisms $\gamma_1 : A_m \to F_1$, $\gamma_2 : B_{m'} \to F_2$ be such that $\psi_1\gamma_1 = \theta_1\varphi_n^m$, $\psi_2\gamma_2 = \theta_2\rho_n^{m'}$. Suppose for instance that $m \leq m'$. Then $(\psi_1 \times \psi_2)((\gamma_1\varphi_m^{m'}) \times \gamma_2) = (\theta_1 \times \theta_2)(\varphi_n^{m'} \times \rho_n^{m'}) : A_{m'} \times B_{m'} \to E_1 \times E_2$. $\square$

So $\mathbb{L} = \mathbb{L}_1 \times \mathbb{L}_2$ is the support of the projective Fraïssé limit of $\mathcal{G}$. Moreover, denoting $\varphi_n : \mathbb{L}_1 \to A_n$, $\rho_n : \mathbb{L}_2 \to B_n$ the projections of the limits onto the members of the fundamental sequences, and given $(a, b), (a', b') \in \mathbb{L}$,

$$(a, b)\ R^{\mathbb{L}}\ (a', b') \Leftrightarrow \forall n \in \mathbb{N}\ (\varphi_n(a), \rho_n(b))\ R^{A_n \times B_n}\ (\varphi_n(a'), \rho_n(b')) \Leftrightarrow$$
$$\Leftrightarrow \forall n \in \mathbb{N}\ (\varphi_n(a)\ R^{A_n}\ \varphi_n(a') \wedge \rho_n(b)\ R^{B_n}\ \rho_n(b')) \Leftrightarrow a\ R^{\mathbb{L}_1}\ a' \wedge b\ R^{\mathbb{L}_2}\ b'.$$

So $\mathbb{L}/R^{\mathbb{L}}$ is homeomorphic to $\mathbb{L}_1/R^{\mathbb{L}_1} \times \mathbb{L}_2/R^{\mathbb{L}_2}$. $\square$

We now prove that the class of finitely representable spaces is closed under taking quotients by $\mathrm{Aut}(\mathbb{L})$-invariant equivalence relations.

Theorem 2.2.4. *Let X be a finitely representable metrizable space; say this is witnessed by a language $\mathcal{L}_R$ and a homeomorphism $\Phi : X \to \mathbb{L}/R^{\mathbb{L}}$. Let $\cong$ be an equivalence relation on X which is such that if $x \cong y$ then $\Phi^{-1}\alpha^*\Phi(x) \cong \Phi^{-1}\alpha^*\Phi(y)$, for all $\alpha \in \mathrm{Aut}(\mathbb{L})$. Then $X/\cong$ is finitely representable.*

Proof. Let $\equiv$ be the closed equivalence relation on $\mathbb{L}$ defined by $u \equiv v$ if and only if $\Phi^{-1}(p(u)) \cong \Phi^{-1}(p(v))$, where $p : \mathbb{L} \to \mathbb{L}/R^{\mathbb{L}}$ is the quotient map. Notice that $\equiv$ extends $R^{\mathbb{L}}$ and that $\alpha^{(2)}[\equiv] = \equiv$, for all $\alpha \in \mathrm{Aut}(\mathbb{L})$.

Let $\mathcal{G}$ be a projective Fraïssé family of finite $\mathcal{L}_R$-structures of which $\mathbb{L}$ is a projective Fraïssé limit.

Claim 2.2.4.1. Assume that $A \in \mathcal{G}$ and let $\psi, \theta : \mathbb{L} \to A$ be $\mathcal{L}_R$-epimorphisms. Then $\psi^{(2)}[\equiv] = \theta^{(2)}[\equiv]$.

Proof of claim. Let $\alpha : \mathbb{L} \to \mathbb{L}$ be an isomorphism such that $\psi = \theta\alpha$. Then $\psi^{(2)}[\equiv] = (\theta\alpha)^{(2)}[\equiv] = \theta^{(2)}[\equiv]$. $\square$

Set $\mathcal{L}'_S = \mathcal{L}_R \cup \{S\}$, where S is a new binary relation symbol. For every $A \in \mathcal{G}$ let A' be the expansion of A to $\mathcal{L}'_S$ defined by letting $S^{A'} = \psi^{(2)}[\equiv]$ for any arbitrary $\mathcal{L}_R$-epimorphism $\psi : \mathbb{L} \to A$. Let $\mathcal{G}' = \{A'\}_{A \in \mathcal{G}}$.

Claim 2.2.4.2. Given $A, B \in \mathcal{G}$, a function $\psi : A \to B$ is an $\mathcal{L}_R$-epimorphism if and only if it is an $\mathcal{L}'_S$-epimorphism from A' to B'.

Proof of claim. The backward implication holds as A', B' are expansions of A, B, respectively.

For the forward direction, it is enough to show that ψ respects S. So let $a, b \in A$ be such that $aS^{A'}b$; pick any $\mathcal{L}_R$-epimorphism $\theta : \mathbb{L} \to A$ and let $u, v \in \mathbb{L}$ be such that $\theta(u) = a$, $\theta(v) = b$, $u \equiv v$. So, by Claim 2.2.4.1, u, v, together with the $\mathcal{L}_R$-epimorphism $\psi\theta : \mathbb{L} \to B$, witness that $\psi(a)S^{B'}\psi(b)$. Conversely, let $a, b \in B$ be such that $aS^{B'}b$ and fix an arbitrary $\mathcal{L}_R$-epimorphism $\theta : \mathbb{L} \to B$; then there are $u, v \in \mathbb{L}$ such that $a = \theta(u)$, $b = \theta(v)$, $u \equiv v$. Let $\gamma : \mathbb{L} \to A$ be an $\mathcal{L}_R$-epimorphism such that $\psi\gamma = \theta$; such an epimorphism exists by combining (L1) and (L3). Then, again by Claim 2.2.4.1, $\gamma(u)S^{A'}\gamma(v)$, $\psi\gamma(u) = a$, $\psi\gamma(v) = b$ and we are done. $\square$

By the claim, $\mathcal{G}'$ is a projective Fraïssé family and a projective Fraïssé limit $\mathbb{L}'$ of $\mathcal{G}'$ is an expansion of $\mathbb{L}$ to $\mathcal{L}'_S$. As for the interpretation of S in $\mathbb{L}'$, we have the following.

Claim 2.2.4.3. $S^{\mathbb{L}'} = \equiv$.

Proof of claim. Let $u, v \in \mathbb{L}'$ and assume first $uS^{\mathbb{L}'}v$. By the closure of $\equiv$, to show $u \equiv v$ it is enough to prove that for any clopen neighbourhoods U, V of u, v, respectively, there are $u' \in U$, $v' \in V$ with $u' \equiv v'$, where we can take $U = V$ if $u = v$, and $U \cap V = \emptyset$ otherwise. So let $A' \in \mathcal{G}'$ with an epimorphism $\psi : \mathbb{L}' \to A'$ refining the clopen covering $\{U, V, \mathbb{L}' \setminus (U \cup V)\}$. Since $\psi(u)S^{A'}\psi(v)$, there are $u', v' \in \mathbb{L}'$ with $\psi(u') = \psi(u)$, $\psi(v') = \psi(v)$, $u' \equiv v'$. Since it follows that $u' \in U, v' \in V$, we are done.

Conversely, suppose $u \equiv v$. Again, fix any clopen neighbourhoods U, V of u, v, respectively, such that $U = V$ if $u = v$, and U, V disjoint otherwise. Pick $A' \in \mathcal{G}'$ and an epimorphism $\psi : \mathbb{L}' \to A'$ refining the clopen covering $\{U, V, \mathbb{L}' \setminus (U \cup V)\}$. Since $\psi(u)S^{A'}\psi(v)$, there are $u', v' \in \mathbb{L}'$ (actually $u' \in U, v' \in V$) with $u'S^{\mathbb{L}'}v'$, and we are done again. $\square$

To finish the proof, notice that $X/\cong$ is homeomorphic to $\mathbb{L}/\equiv$. $\square$

Remark 2.2.5. In the setting of the previous result, let $\sim$ be an equivalence relation on X which is the identity outside $\Phi[p[G]]^2$, where $G \subseteq \mathbb{L}$ is the set of almost stable points. If $x \sim y$, then $\Phi^{-1}\alpha^*\Phi(x) \sim \Phi^{-1}\alpha^*\Phi(y)$, for all $\alpha \in \mathrm{Aut}(\mathbb{L})$, by definition of almost stable. So $X/\sim$ is finitely representable.

2.3 Arcs, hypercubes, graphs

We now apply the results of the preceding sections to demonstrate the finite representability of some classes of continua. We begin by establishing the following.

Theorem 2.3.1. *Arcs are finitely representable.*

We prove Theorem 2.3.1 through a sequence of lemmas.

Let $\mathcal{L}_R = \{R, \leq\}$, where $\leq$ is a binary relation symbol. Let $\mathcal{X}$ be the class [3] of those finite $\mathcal{L}_R$-structures A such that:

- $\leq^A$ is a total order;
- $a\, R^A\, b$ if and only if $a = b$ or a, b are $\leq^A$-consecutive.

Lemma 2.3.2. *Class $\mathcal{X}$ is a projective Fraïssé family.*

Proof. If $A = \{1\} \in \mathcal{X}$ is defined by letting $R^A = \leq^A = \{(1,1)\}$, then for any $B \in \mathcal{X}$ the constant map $\psi : B \to A$ is an epimorphism. So it is enough to verify (AP), since together with the existence of a final object in $\mathcal{X}$, it implies (JPP).

Let $A, B, C \in \mathcal{X}$ with epimorphisms $\psi : B \to A$, $\theta : C \to A$. Let

$$a_1 \leq^A \ldots \leq^A a_{|A|}$$

be an enumeration of A. Let $N_j = \max\{|\psi^{-1}(a_j)|, |\theta^{-1}(a_j)|\}$, for each $j \in \{1, \ldots, |A|\}$, and define $D \in \mathcal{X}$ such that

$$|D| = \sum_{j=1}^{|A|} N_j$$

and enumerate it as $D = \{d_{jl} \mid j \in \{1, \ldots, |A|\}, l \in \{1, \ldots, N_j\}\}$. Let $\leq^D$ be the total order on D determined by the lexicographic order on the pairs of indices (j, l). This determines relation R^D too.

Now define $\gamma : D \to B$ by mapping $\{d_{j1}, \ldots, d_{jN_j}\}$ onto $\psi^{-1}(a_j)$ in an increasing way, and similarly define $\rho : D \to C$. So γ, ρ are epimorphisms and $\psi\gamma = \theta\rho$. $\square$

Let $\mathbb{X}$ be the projective Fraïssé limit of $\mathcal{X}$.

[3] This class coincides with that of finite HLO's, or Hasse linear orders, which appears in Chapter 3.

Lemma 2.3.3. *The relation $\leq^{\mathbb{X}}$ is a total order on $\mathbb{X}$ having a least and a last element.*

Proof. By Proposition 2.1.1, parts (1)(3)(4)(5)(6). $\square$

Lemma 2.3.4. *The relation $R^{\mathbb{X}}$ is an equivalence relation.*

Proof. By Proposition 2.1.1, parts (1)(2), $R^{\mathbb{X}}$ is reflexive and symmetric. To complete the proof, it will be shown that every $u \in \mathbb{X}$ is $R^{\mathbb{X}}$-related to at most one element different from itself.

So suppose, towards a contradiction, that u, v_1, v_2 are distinct elements in $\mathbb{X}$ such that $v_1 \; R^{\mathbb{X}} \; u \; R^{\mathbb{X}} \; v_2$. Let U, V_1, V_2 be disjoint clopen neighbourhoods of u, v_1, v_2, respectively. If $\psi : \mathbb{X} \to A$ is any epimorphism onto an element of $\mathcal{X}$ refining $\{U, V_1, V_2, \mathbb{X} \setminus (U \cup V_1 \cup V_2)\}$, since $\psi(u)$, $\psi(v_1)$, $\psi(v_2)$ are distinct and $\psi(v_1) \; R^A \; \psi(u) \; R^A \; \psi(v_2)$, it follows that $\psi(v_1), \psi(u), \psi(v_2)$ are $\leq^A$-consecutive, with $\psi(u)$ being the midpoint. Say, for instance, $\psi(v_1) \leq^A \psi(u) \leq^A \psi(v_2)$. Then let $B = A \cup \{z\}$, where $z \notin A$, with the symbols of $\mathcal{L}_R$ interpreted as follows:

- $\leq^B$ is obtained from $\leq^A$ by inserting z between $\psi(u), \psi(v_2)$;
- $a \; R^B \; b$ if and only if $a = b$ or a, b are $\leq^B$-consecutive.

So $B \in \mathcal{X}$. Define $\theta : B \to A$ as the identity on the elements of A and by letting $\theta(z) = \psi(u)$. Then there cannot be any epimorphism $\gamma : \mathbb{X} \to B$ such that $\psi = \theta\gamma$, since $\gamma(u)$ could not be R^B-related to both $\psi(v_1), \psi(v_2)$. $\square$

Lemma 2.3.5. *If $u \in \mathbb{X}$ then u has a basis of clopen neighbourhoods that are convex sets with respect to $\leq^{\mathbb{X}}$.*

Proof. Let U be a clopen subset of $\mathbb{X}$ containing u. Let $\psi : \mathbb{X} \to A$ be an epimorphism onto some $A \in \mathcal{X}$ refining the clopen covering $\{U, \mathbb{X} \setminus U\}$. Let $V = \psi^{-1}(\psi(u))$, so that V is clopen. If $v, z \in V$ with $v \leq^{\mathbb{X}} z$, then for any $w \in \mathbb{X}$ with $v \leq^{\mathbb{X}} w \leq^{\mathbb{X}} z$ one has $\psi(w) = \psi(u)$, whence $w \in V$. $\square$

Lemma 2.3.6. *If $u, v \in \mathbb{X}$, then u, v are $\leq^{\mathbb{X}}$-consecutive if and only if they are distinct and $R^{\mathbb{X}}$-related.*

Proof. Suppose $u \leq^{\mathbb{X}} v$, so that in particular $\psi(u) \leq^A \psi(v)$ for any epimorphism ψ from $\mathbb{X}$ onto some $A \in \mathcal{X}$.

Assume first they are consecutive (in particular, $u \neq v$). First, notice that for any $A \in \mathcal{X}$ and epimorphism $\psi : \mathbb{X} \to A$ either $\psi(u) = \psi(v)$ or $\psi(u), \psi(v)$ are $\leq^A$-consecutive, since ψ is monotone with respect to the orders. So it follows that $\psi(u) \, R^A \, \psi(v)$. By the arbitrariness of A and ψ, this implies $u \, R^{\mathbb{X}} \, v$.

Conversely, assume $u \neq v$, $u R^{\mathbb{X}} v$ and suppose there is $z \in \mathbb{X}$ with $u <^{\mathbb{X}} z <^{\mathbb{X}} v$. Let U, V, W be disjoint clopen neighbourhoods of u, v, z, respectively. Let $A \in \mathcal{X}$ with an epimorphism $\psi : \mathbb{X} \to A$ refining $\{U, V, W, \mathbb{X} \setminus (U \cup V \cup W)\}$. Then $\psi(u) <^A \psi(z) <^A \psi(v)$, so $\psi(u), \psi(v)$ are not R^A-related, a contradiction. $\square$

Lemma 2.3.7. *A closed total order $\leq$ on a compact metric space X is complete.*

Proof. Let A be a bounded non-empty subset of X. Let $A' = \{x \in X \mid \forall y \in A\ y \leq x\}$, the set of upper bounds of A, which is a closed non-empty subset of X. It is then enough to establish the existence of $\min A'$. Let $\{x_\alpha\}_{\alpha\in\beta}$ be a maximal decreasing sequence in A'. Since every $\leq$-open interval is an open subset of X, by separability of X the ordinal β must be countable. If $\beta = \gamma + 1$ is a successor ordinal, then $x_\gamma = \min A'$. Otherwise, by compactness, $\inf\{x_\alpha\}_{\alpha\in\beta}$ exists and it equals $\min A'$. □

Let $I = \mathbb{X}/R^{\mathbb{X}}$ and let $\varphi : \mathbb{X} \to I$ be the quotient map. On I define $[u] \leq^I [v]$ if and only if $u \leq^{\mathbb{X}} v$. By Lemma 2.3.6 this is well defined. Moreover, by Lemmas 2.3.3, 2.3.6 and 2.3.7, this is a dense, complete total order with a first and a last element.

Lemma 2.3.8. *The quotient topology on I is the order topology induced by $\leq^I$.*

Proof. We first show that sets of the form $I_{[a]} = \{[u] \in I \mid [a] <' [u]\}$, $I^{[b]} = \{[u] \in I \mid [u] <' [b]\}$ are open in I. For the first kind, since $[a]$ contains at most two elements, let a^* be its maximum with respect to $\leq^{\mathbb{X}}$. Then $I_{[a]}$ is the image under φ of $\{u \in \mathbb{X} \mid a^* <^{\mathbb{X}} u\}$, which is open (since $\leq^{\mathbb{X}}$ is closed and total) and $R^{\mathbb{X}}$-invariant. The same argument works for the second type of intervals.

Conversely, let U be open in I and fix $[u] \in U$. By Lemma 2.3.5 for each point in $[u]$ there is a $\leq^{\mathbb{X}}$-convex, clopen subset of $\mathbb{X}$ containing that point and included in $\varphi^{-1}(U)$. Since $[u]$ is either a singleton or a doubleton consisting of two $\leq^{\mathbb{X}}$-consecutive points, the union of these clopen sets, call it I, is $\leq^{\mathbb{X}}$-convex. It is then enough to show that, if $\min I \neq [u]$, then I contains some element that strictly precedes all elements of $[u]$, and similarly that if $\max I \neq [u]$ then I contains some element strictly bigger than the elements of $[u]$. So suppose $\min I \neq [u]$. If, towards a contradiction, $[u]$ contained the least element of I, let J be the set of all strict predecessors of $\min I$. Since I is clopen and $\leq^{\mathbb{X}}$ is closed, J is a clopen, non-empty, bounded subset of $\mathbb{X}$. By Lemma 2.3.7, J has a maximum z. So z is an immediate predecessor of $\min I$, but z and $\min I$ are not $R^{\mathbb{X}}$-related, since $\min I \in [u] \subseteq I$. This contradicts Lemma 2.3.6. □

Lemma 2.3.9. *$\leq^I$ has order type $1 + \lambda + 1$, where λ is the order type of the real line.*

Proof. We already noted that $\leq^I$ is bounded and complete. We remark that it is also a separable order: indeed, it is a dense order, so every open interval is non-empty and, by Lemma 2.3.8, open in the Polish space I, thus every interval contains a point of a fixed countable dense subset of I. Since by [Ros82, Theorem 2.30] a separable and complete total order without first or last element has order type λ, we are done. □

Since the topology of I is induced by an order of type $1 + \lambda + 1$, it follows that I is an arc, concluding the proof of Theorem 2.3.1. Notice that $\mathrm{Aut}(I)$ is the subgroup of homeomorphisms of the arc which preserve $\leq^I$.

An immediate consequence is now the following. Recall that a hypercube is a space homeomorphic to $[0, 1]^n$, for some n.

Corollary 2.3.10. *Every hypercube is finitely representable.*

Proof. By Theorems 2.2.3 and 2.3.1. □

For the next consequence recall that, in continuum theory, a graph is defined as a finite union of arcs any two of them meeting at most in one or both of their endpoints (see for example [Nad92]).

Corollary 2.3.11. *Every graph is finitely representable.*

Proof. Notice that in the proof of Theorem 2.3.1 each endpoint of arc I is the image under the quotient map of an almost stable point, since the extrema of a total order — in this case $\leq^{\mathbb{X}}$ — are preserved under isomorphism. So we can use Theorem 2.2.2 to obtain a disjoint union of arcs; the remark following that theorem allows us to apply Theorem 2.2.4 to glue endpoints and thus obtain any possible graph. □

2.4 Questions

In the previous sections we exhibited some simple classes of finitely representable spaces, enlarging the examples given in [Cam10]. This suggests the following general question.

Question 2.4.1. What spaces are finitely representable?

In our examples, due to the application of the constructions of Section 2.2, the languages and the structures associated to the spaces were in some sense always related to the obvious structural characteristics of the spaces, starting from an order representing the arc. The following rather vague question comes to mind.

Question 2.4.2. Given a finitely representable space, what are the minimal, or most natural, language and structures representing it? Can some specific features of the space be derived directly from the language? What are the obstructions that forbid a space to be represented with a given language?

Chapter 3

Smooth fences

In this chapter we introduce and begin the study of a new class of topological spaces, which we call *fences*. Among them, we define the subclass of smooth fences and characterize them as those fences admitting an embedding in $2^{\mathbb{N}} \times [0,1]$. We relate smooth fences to a class of finite structures.

3.1 Finite Hasse forests

Henceforth fix $\mathcal{L}_R = \{R, \leq\}$, where $\leq$ is a binary relation symbol. A *Hasse partial order* (*HPO*) is a compact metrizable $\mathcal{L}_R$-structure P such that

- $\leq^P$ is a partial order, that is, it is reflexive, anti-symmetric and transitive;
- $a \, R^P \, b$ if and only if $a = b$ or a, b are $\leq^P$-consecutive, that is:
  - $a \leq^P b$ and whenever $a \leq^P c \leq^P b$ it holds that $c = a$ or $c = b$; or
  - $b \leq^P a$ and whenever $b \leq^P c \leq^P a$ it holds that $c = a$ or $c = b$.

Indeed, if P is a finite HPO, the relation R^P is the Hasse diagram [1] of $\leq^P$. Where clear we shall write $a \leq b$ instead of $a \leq^P b$, and similarly for $a < b$ and $a \, R \, b$. When $a \leq b$ we also let $[a, b] = \{c \in P \mid a \leq c \leq b\}$. If $\leq^P$ is total, then we say that P is a *Hasse linear order* (or *HLO*).

If P, P' are HPOs we denote by $P \sqcup P'$ the HPO where the support and the interpretations of $\leq$ and R are the disjoint unions of the corresponding notions in P, P'.

Definition 3.1.1. A *Hasse forest* (*H-forest*) is a HPO whose Hasse diagram has no cycles, and we denote by $\mathcal{F}$ the family of all finite H-forests.

Definition 3.1.2. For an HPO P, denote by $\mathrm{MC}(P)$ the set of maximal chains of P with respect to the partial order $\leq^P$.

[1] see [GYZ14, Section 3.2.3] for the definition and examples.

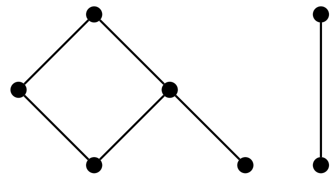

Figure 3.1: A representation of a finite HPO P. The edges represent the relation R^P and $a \leq^P b$ if and only if a is connected to b by a (possibly empty) sequence of ascending edges.

Notice that if $P \in \mathcal{F}$ and $B \in \mathrm{MC}(P)$ then B is the unique maximal chain to which both $\min B$ and $\max B$ belong. Indeed, if $B' \in \mathrm{MC}(P)$ is such that $\min B, \max B \in B'$ then $\min B' = \min B$ and $\max B' = \max B$ by the maximality of B, so if $B \neq B'$ there would be two R^P-paths joining $\min B$ and $\max B$.

In [BK15] it is shown[2] that the class of all finite H-forests with a minimum is a projective Fraïssé family whose limit's quotient with respect to R is the Lelek fan. In Lemma 2.3.2 it is shown that the class $\mathcal{X}$ of all finite HLOs is a projective Fraïssé family whose limit's quotient is the arc. Here we prove that, though the family of all finite HPOs is not a projective Fraïssé family, the family of all finite H-forests is.

We begin by describing a smaller yet cofinal family which plays a central role in the rest of this dissertation.

Definition 3.1.3. Let $\mathcal{F}_0$ be the collection of all $P \in \mathcal{F}$ whose maximal chains are pairwise disjoint. In other words, the elements of $\mathcal{F}_0$ are the finite disjoint unions of finite HLOs.

Notice that if $P \in \mathcal{F}_0$ and $Q \subseteq P$ is $\leq^P$-convex — that is, whenever $b, b' \in Q, a \in P$ are such that $b \leq^P a \leq^P b'$, then $a \in Q$ — then Q with the induced $\mathcal{L}_R$-structure is in $\mathcal{F}_0$.

Proposition 3.1.4. *$\mathcal{F}_0$ is cofinal in the family of all finite HPOs.*

Proof. Let P be a finite HPO. If $\mathrm{MC}(P) = \{B_1, \ldots, B_m\}$, let $P' = B'_1 \sqcup \ldots \sqcup B'_m$ where every B'_j is isomorphic to B_j with the induced structure. Then there is an epimorphism $\varphi : P' \to P$, given by letting φ be an isomorphism from B'_j onto B_j for $1 \leq j \leq m$. $\square$

Proposition 3.1.5. *The family of all finite HPOs is not a projective Fraïssé family.*

[2] Albeit with a different language, it is easy to see that a continuous surjection is an epimorphism with one such language iff it is so with the other, thus ensuring that the limit is the same.

Proof. We show that the family of all finite HPOs lacks amalgamation. Let

$$\begin{aligned} S =& \{a, b, c, d\}, \\ P =& \{a_0, b_0, b_0', c_0, d_0\}, \\ Q =& \{a_1, b_1, c_1, c_1', d_1\}, \end{aligned}$$

be ordered as follows (see Figure 3.2):

- For S: $a = \min S, d = \max S$, and b, c are incomparable.
- For P: $a_0 < b_0, a_0 < c_0 < d_0, b_0' < d_0$, and no other order comparabilities hold, except for reflexivity and transitivity.
- For Q: $a_1 < b_1 < d_1, a_1 < c_1, c_1' < d_1$, and no other order comparabilities hold, except for reflexivity and transitivity.

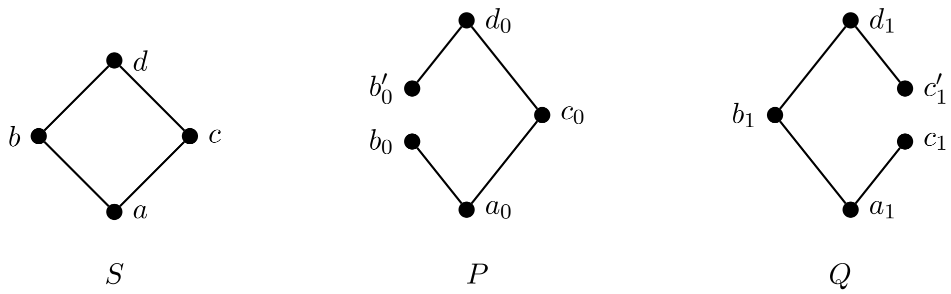

Figure 3.2

Define $\varphi : P \to S, \psi : Q \to S$ by letting:

$$\begin{aligned} &\varphi(a_0) = \psi(a_1) = a, \\ &\varphi(b_0) = \varphi(b_0') = \psi(b_1) = b, \\ &\varphi(c_0) = \psi(c_1) = \psi(c_1') = c, \\ &\varphi(d_0) = \psi(d_1) = d. \end{aligned}$$

Then φ, ψ are epimorphisms. To show that there is no amalgamation, by Proposition 3.1.4 it is enough to show that there is no $F \in \mathcal{F}_0$ with epimorphisms $\theta : F \to P, \rho : F \to Q$ such that $\varphi\theta = \psi\rho$. Otherwise, as $a_0 < d_0$, there must be $B \in \mathrm{MC}(F)$ and $i, i' \in B$, with $i < i'$, such that $\theta(i) = a_0, \theta(i') = d_0$, so that $\theta[B] = \{a_0, c_0, d_0\}$; moreover $\rho(i) = a_1, \rho(i') = d_1$. If $j \in B$ is such that $\theta(j) = c_0$, then $i < j < i'$ and $\rho(j) \in \{c_1, c_1'\}$, since $\varphi\theta = \psi\rho$. If $\rho(j) = c_1$, this contradicts $j \leq i'$, as $\rho(j) \not\leq \rho(i')$; similarly, if $\rho(j) = c_1'$, this contradicts $i \leq j$. $\square$

Let us turn to the proof of the central result of the section.

Theorem 3.1.6. *The family $\mathcal{F}$ of all finite H-forests is a projective Fraïssé family.*

First, we note the following simple but useful observation.

Lemma 3.1.7. *Let $P, P' \in \mathcal{F}$, and let $\varphi : P \to P'$ be an epimorphism. If $B \in \mathrm{MC}(P)$, then there is $B' \in \mathrm{MC}(P')$ such that $\varphi[B] \subseteq B'$. If $B' \in \mathrm{MC}(P')$, then there exists $B \in \mathrm{MC}(P)$ such that $\varphi[B] = B'$.*

Proof. For the first statement, since $B \in \mathrm{MC}(P)$ and φ is an epimorphism, then $\varphi[B]$ is a chain in P', so $\varphi[B]$ is included in a maximal chain.

For the second assertion, fix $B' \in \mathrm{MC}(P')$. Since $\min B' \leq \max B'$ and φ is an epimorphism, there are $a, b \in P$ such that $a \leq b, \varphi(a) = \min B', \varphi(b) = \max B'$. Let $B \in \mathrm{MC}(P)$ contain a, b. Since $\min B \leq a$ then $\varphi(\min B) \leq \min B'$, so $\varphi(\min B) = \min B'$; analogously, $\varphi(\max B) = \max B'$. Since P' is an H-forest and φ respects R, it follows that $\varphi[B] = B'$. $\square$

We can also prove a sort of converse. Given $\mathcal{L}_R$-structures P, P' and a function $\varphi : P \to P'$, we say that φ is *$\mathcal{L}_R$-preserving* if $a \, R^P \, b \Rightarrow \varphi(a) \, R^{P'} \, \varphi(b)$ and $a \leq^P b \Rightarrow \varphi(a) \leq^{P'} \varphi(b)$, for every $a, b \in P$.

Lemma 3.1.8. *Let $P, P' \in \mathcal{F}$, and let $\varphi : P \to P'$ be an $\mathcal{L}_R$-preserving function. If for each $B' \in \mathrm{MC}(P')$ there exists $B \in \mathrm{MC}(P)$ such that $\varphi[B] = B'$, then φ is an epimorphism.*

Proof. The function φ is clearly surjective. Let $a', b' \in P'$ such that $a' \leq b'$ and let $B' \in \mathrm{MC}(P')$ with $a', b' \in B'$. Let $B \in \mathrm{MC}(P)$ such that $\varphi[B] = B'$, then there are $a, b \in B$ such that $\varphi(a) = a', \varphi(b) = b'$ and $a \leq b$. If $a' \, R \, b'$ with $a' < b'$, then a, b can be chosen to be R^P-related by letting $a = \max(B \cap \varphi^{-1}(a'))$ and $b = \min(B \cap \varphi^{-1}(b'))$. $\square$

Proof of Theorem 3.1.6. Since for every $P \in \mathcal{F}$ there is an epimorphism from P to the H-forest consisting of a single point, it suffices to prove amalgamation. Let $P, Q, S \in \mathcal{F}$ and epimorphisms $\varphi : P \to S$, $\psi : Q \to S$.

For each $C \in \mathrm{MC}(P)$, by Lemma 3.1.7 there is $D \in \mathrm{MC}(Q)$ such that $\psi[D] \supseteq \varphi[C]$. Let $C' = \psi^{-1}(\varphi[C]) \cap D$. Since $C, \varphi[C], C'$ with the inherited relations are finite HLOs and $\varphi{\restriction_C}, \psi{\restriction_{C'}}$ are, in particular, epimorphisms onto $\varphi[C]$, by Lemma 2.3.2, there exist $E_C \in \mathcal{X}$ and epimorphisms $\varphi'_C : E_C \to C$, $\psi'_C : E_C \to C'$ and such that $\varphi{\restriction_C}\varphi'_C = \psi{\restriction_{C'}}\psi'_C$.

Analogously, for each $C \in \mathrm{MC}(Q)$ there exists $D \in \mathrm{MC}(P)$ such that $\varphi[D] \supseteq \psi[C]$. As above there exist $E_C \in \mathcal{X}$ and epimorphisms $\varphi'_C : E_C \to C' = \varphi^{-1}(\psi[C]) \cap D$ and $\psi'_C : E_C \to C$ such that $\varphi{\restriction_{C'}}\varphi'_C = \psi{\restriction_C}\psi'_C$.

Define the $\mathcal{L}_R$-structure:

$$T = \bigsqcup \{E_C \mid C \in \mathrm{MC}(P) \sqcup \mathrm{MC}(Q)\} \in \mathcal{F}_0,$$

and $\varphi' : T \to P, \psi' : T \to Q$, where, for $x \in E_C$, $\varphi'(x) = \varphi'_C(x)$ and $\psi'(x) = \psi'_C(x)$. By construction $\varphi\varphi' = \psi\psi'$. Since φ'_C, ψ'_C are epimorphisms then φ', ψ' are $\mathcal{L}_R$-preserving. Let $C \in \mathrm{MC}(P)$, then $\varphi'[E_C] = \varphi'_C[E_C] = C$. Analogously if $C \in \mathrm{MC}(Q)$, then $\psi'[E_C] = \psi'_C[E_C] = C$. By Lemma 3.1.8, φ', ψ' are thus epimorphisms. $\square$

By Theorem 3.1.6, Proposition 3.1.4 and Proposition 1.1.1 it follows that:

Corollary 3.1.9. *$\mathcal{F}_0$ is a projective Fraïssé family with the same projective Fraïssé limit as $\mathcal{F}$.*

3.2 Projective limits of sequences in $\mathcal{F}_0$

In the next section we determine the spaces which are approximable by fine projective sequences from $\mathcal{F}_0$. For this, we establish some properties of projective sequences in $\mathcal{F}_0$ and their limits which are of use later. For the remainder of the section let (P_n, φ_n^m) be a fine projective sequence in $\mathcal{F}_0$ with projective limit $\mathbb{P}$, and $p : \mathbb{P} \to \mathbb{P}/R^{\mathbb{P}}$ be the quotient map. Notice that $\leq^{\mathbb{P}}$ is a partial order relation.

Lemma 3.2.1. *Let $u, v \in \mathbb{P}$ with $u \leq v$. Then $[u, v]$ is R-connected.*

Proof. First notice that the sequence $\varphi_n^{-1}([\varphi_n(u), \varphi_n(v)])$ converges in $\mathcal{K}(\mathbb{P})$ to $[u, v]$, since $\forall n \in \mathbb{N}\ \varphi_{n+1}^{-1}([\varphi_{n+1}(u), \varphi_{n+1}(v)]) \subseteq \varphi_n^{-1}([\varphi_n(u), \varphi_n(v)])$ and

$$\bigcap_{n \in \mathbb{N}} \varphi_n^{-1}([\varphi_n(u), \varphi_n(v)]) = [u, v].$$

By Lemma 1.2.10 it is now enough to observe that every $[\varphi_n(u), \varphi_n(v)]$ is R-connected. $\square$

Lemma 3.2.2. *The $R^{\mathbb{P}}$-equivalence classes contain at most two elements; moreover, each class is totally ordered and convex with respect to $\leq^{\mathbb{P}}$.*

Proof. Let $u, v, w \in \mathbb{P}$ be $R^{\mathbb{P}}$-related elements. If u, v, w were all distinct, there would exist $n \in \mathbb{N}$ such that $\varphi_n(u), \varphi_n(v), \varphi_n(w)$ are all distinct and pairwise R^{P_n}-related, which is impossible, since $P_n \in \mathcal{F}_0$.

If $u\ R^{\mathbb{P}}\ v$, then $\varphi_n(u)\ R^{P_n}\ \varphi_n(v)$ for every n; in particular, $\varphi_n(u), \varphi_n(v)$ are $\leq^{P_n}$ comparable for every n. It follows that either $\forall n \in \mathbb{N}\ \varphi_n(u) \leq \varphi_n(v)$ or $\forall n \in \mathbb{N}\ \varphi_n(v) \leq \varphi_n(u)$, whence either $u \leq v$ or $v \leq u$.

Finally, if $u\ R^{\mathbb{P}}\ v$ but $u <^{\mathbb{P}} w <^{\mathbb{P}} v$ for some $u, v, w \in \mathbb{P}$, let $n \in \mathbb{N}$ be such that $\varphi_n(u), \varphi_n(v), \varphi_n(w)$ are distinct. Then both $\varphi_n(u)\ R^{P_n}\ \varphi_n(v)$ and $\varphi_n(u) <^{P_n} \varphi_n(w) <^{P_n} \varphi_n(v)$, which is a contradiction. $\square$

Lemma 3.2.3. *If $u, v \in \mathbb{P}$ are not $R^{\mathbb{P}}$-related and $u \leq v$ holds, then whenever $u'\ R^{\mathbb{P}}\ u, v'\ R^{\mathbb{P}}\ v$, the relation $u' \leq^{\mathbb{P}} v'$ holds.*

Proof. For $n \in \mathbb{N}$ big enough, $\varphi_n(u), \varphi_n(v)$ are distinct and not R^{P_n}-related. Since $\varphi_n(u) \leq \varphi_n(v)$, $P_n \in \mathcal{F}_0$, and R^{P_n}-related distinct elements are one the immediate $\leq^{P_n}$-successor of the other and viceversa, it follows that $\varphi_n(u') \leq \varphi_n(v')$. This inequality holding eventually, the relation $u' \leq^{\mathbb{P}} v'$ is established. □

Lemma 3.2.4. *The maximal chains of $\mathbb{P}$ are pairwise disjoint.*

Proof. Suppose that $u, v, v' \in \mathbb{P}$ are such that u is $\leq^{\mathbb{P}}$-comparable to v, v'. For each $n \in \mathbb{N}$, $\varphi_n(u)$ is $\leq^{\mathbb{P}}$-comparable to $\varphi_n(v), \varphi_n(v')$. But $P_n \in \mathcal{F}_0$, so $\varphi_n(v), \varphi_n(v')$ are $\leq^{P_n}$-comparable. It follows that v, v' are $\leq^{\mathbb{P}}$-comparable. □

Corollary 3.2.5. *The relation $\leq^{\mathbb{P}/R^{\mathbb{P}}} = p^{(2)}[\leq^{\mathbb{P}}]$ on $\mathbb{P}/R^{\mathbb{P}}$ is a closed order relation.*

Proof. That $\leq^{\mathbb{P}/R^{\mathbb{P}}}$ is closed is observed at the beginning of Section 1.2. Moreover:

- $\leq^{\mathbb{P}/R^{\mathbb{P}}}$ is reflexive by the reflexivity of $\leq^{\mathbb{P}}$.
- If $x \leq^{\mathbb{P}/R^{\mathbb{P}}} y \leq^{\mathbb{P}/R^{\mathbb{P}}} z$ with $x \neq y \neq z$, let

$$\begin{aligned} &u \in p^{-1}(x), \\ &v, v' \in p^{-1}(y), \\ &w \in p^{-1}(z), \end{aligned}$$

  with $u \leq^{\mathbb{P}} v, v' \leq^{\mathbb{P}} w$; by Lemma 3.2.3 it follows that $u \leq^{\mathbb{P}} v'$, so that $u \leq^{\mathbb{P}} w$ and finally $x \leq^{\mathbb{P}/R^{\mathbb{P}}} z$.
- If $x \leq^{\mathbb{P}/R^{\mathbb{P}}} y \leq^{\mathbb{P}/R^{\mathbb{P}}} x$, there are

$$\begin{aligned} &u, u' \in p^{-1}(x), \\ &v, v' \in p^{-1}(y), \end{aligned}$$

  with $u \leq^{\mathbb{P}} v, v' \leq^{\mathbb{P}} u'$; by Lemma 3.2.2 it follows that $u\, R^{\mathbb{P}}\, v$, and finally $x = y$.

□

Lemma 3.2.6. *If $B \in \mathrm{MC}(P_n)$ then $\bigcup_{a \in B} [\![a]\!]_{\varphi_n}$ is a clopen subset of $\mathbb{P}/R^{\mathbb{P}}$.*

Proof. Since for each $a \in B$ the set $\varphi_n^{-1}(a)$ is clopen, it follows that $\bigcup_{a \in B} \varphi_n^{-1}(a)$ is clopen. Let $u, v \in \mathbb{P}$ be such that $u \in \bigcup_{a \in B} \varphi_n^{-1}(a)$ and $u R^{\mathbb{P}} v$. Then $\varphi_n(u)\, R^{P_n}\, \varphi_n(v)$, so $\varphi_n(v) \in B$, that is, $v \in \bigcup_{a \in B} \varphi_n^{-1}(a)$. It follows that $\bigcup_{a \in B} \varphi_n^{-1}(a)$ is $R^{\mathbb{P}}$-invariant, so $\bigcup_{a \in B} [\![a]\!]_{\varphi_n} = p[\bigcup_{a \in B} \varphi_n^{-1}(a)]$ is open, thus clopen. □

A converse of the above also holds.

Lemma 3.2.7. *Let C be a clopen subset of $\mathbb{P}/R^{\mathbb{P}}$. There is $n \in \mathbb{N}$ such that for all $m \geq n$, there is $S \subseteq \mathrm{MC}(P_m)$ for which $C = \bigcup_{a \in \bigcup S} [\![a]\!]_{\varphi_m}$.*

Proof. First notice that it is enough to show that there are some $n \in \mathbb{N}$ and $S \subseteq \mathrm{MC}(P_n)$ for which $C = \bigcup_{a \in \bigcup S} [\![a]\!]_{\varphi_n}$. Indeed, assuming this, let $m \geq n$. Then $(\varphi_n^m)^{-1}(\bigcup S) = \bigcup T$ for some $T \subseteq \mathrm{MC}(P_m)$, and $C = \bigcup_{a \in \bigcup T} [\![a]\!]_{\varphi_m}$.

Since $p^{-1}(C)$ is compact and open and the sets $\{\varphi_n^{-1}(a) \mid n \in \mathbb{N}, a \in A_n\}$ form a basis for the topology of $\mathbb{P}$, there exist $n \in \mathbb{N}$ and a subset $B \subseteq P_n$ such that $p^{-1}(C) = \bigcup_{a \in B} \varphi_n^{-1}(a)$, so that $B = \varphi_n[p^{-1}(C)]$.

We prove that $B = \bigcup S$ for some $S \subseteq \mathrm{MC}(P_n)$. If this were not the case, there would exist $a, a' \in P_n$ with a, a' consecutive with respect to $\leq^{P_n}$ and $a \in B, a' \notin B$; in particular, $a \, R \, a'$. If $u, u' \in \mathbb{P}$ are such that $\varphi_n(u) = a, \varphi_n(u') = a', u \, R \, u'$, then $u \in p^{-1}(C), u' \notin p^{-1}(C)$ contradict the fact that $p^{-1}(C)$ is $R^{\mathbb{P}}$-invariant. The proof is concluded by observing that:

$$C = p(p^{-1}[C]) = p[\bigcup_{a \in B} \varphi_n^{-1}(a)] = \bigcup_{a \in B} [\![a]\!]_{\varphi_n}.$$

□

3.3 Fences

Definition 3.3.1. A *fence* is a compact metrizable space whose connected components are either points or arcs. A fence Y is *smooth* if there is a closed partial order $\preceq$ on Y whose restriction to each connected component of Y is a total order.

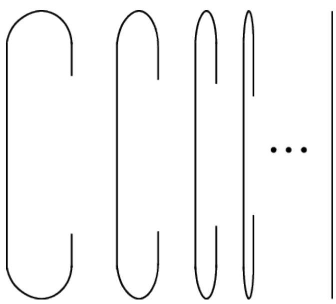

Figure 3.3: A fence which is not smooth: a sequence of crooked arcs converging to a straight arc in a fashion remeniscent of a staple being deformed by the stapler.

We call *arc components* of a fence the connected components which are arcs, and *singleton components* those which are points. Recall that a point x in a topological space X is an *endpoint* if whenever x belongs to an arc $[a, b] \subseteq X$, then $x = a$ or $x = b$ (note that under this definition singleton components are endpoints). We denote by $\mathrm{E}(Y)$ the set of endpoints of a fence Y; equivalently, $\mathrm{E}(Y)$ is the set of endpoints of the connected components of Y. The *Cantor fence* is the space $2^{\mathbb{N}} \times [0, 1]$; it is a smooth fence, as witnessed by the product of equality on $2^{\mathbb{N}}$ and the usual ordering of $[0, 1]$: we denote this order by $\trianglelefteq$.

Theorem 3.3.2 below establishes that smooth fences are, up to homeomorphism, the compact subspaces of the Cantor fence. It may be confronted with [CC89, Proposition 4], stating that smooth fans are, up to homeomorphism, the subcontinua of the Cantor fan, which is the fan obtained by identifying in the Cantor fence the set $2^{\mathbb{N}} \times \{0\}$ to a point.

Recall that if X is a topological space and $f : X \to [0,1]$ is a function, then f is *lower semi-continuous* (l.s.c.) if $\{x \in X \mid f(x) \leq y\}$ is closed for each $y \in [0,1]$ and is *upper semi-continuous* (u.s.c.) if $\{x \in X \mid f(x) \geq y\}$ is closed for each $y \in [0,1]$.

Let X be a zero-dimensional, compact, metrizable space and $m, M : X \to [0,1]$ be two functions. We say that (m, M) is a *fancy pair* if

- m is l.s.c.;
- M is u.s.c.;
- $m(x) \leq M(x)$, for all $x \in X$.

If (m, M) is a fancy pair of functions on X, let $D_m^M = \{(x,y) \in X \times [0,1] \mid m(x) \leq y \leq M(x)\}$. Then D_m^M is a closed subset of $X \times [0,1]$. Indeed, let $(x_n, y_n) \in D_m^M$, and $(x,y) = \lim(x_n, y_n)$. Then for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $m > n$,

$$m(x) - \varepsilon < m(x_m) \leq y_n \leq M(x_m) < M(x) + \varepsilon,$$

so $m(x) \leq y \leq M(x)$, thus $(x,y) \in D_m^M$.

Theorem 3.3.2. *Let Y be a fence. Then the following are equivalent:*

1. *Y is a smooth fence.*
2. *There exists a closed partial order $\preceq$ on Y whose restriction to each connected component is a total order and such that two elements are $\preceq$-comparable if and only if they belong to the same connected component.*
3. *There is a continuous injection $f : Y \to 2^{\mathbb{N}} \times [0,1]$.*
4. *There is a continuous injection $f : Y \to 2^{\mathbb{N}} \times [0,1]$ such that for each $x \in 2^{\mathbb{N}}$, the set $f[Y] \cap (\{x\} \times [0,1])$ is connected (possibly empty).*
5. *There is a closed, non-empty, subset X of $2^{\mathbb{N}}$ and a fancy pair (m, M) of functions on X such that Y is homeomorphic to D_m^M.*

Proof. The implications (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3) are immediate. The implications (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) follow by copying on Y the restriction of the order $\trianglelefteq$ on the Cantor fence to the image of Y under the embedding.

For (4) $\Rightarrow$ (5), let $X = \pi_1[f[Y]]$ be the projection of $f[Y]$ on $2^{\mathbb{N}}$ and, for $x \in X$, let $m(x) = \min\{y \in [0,1] \mid (x,y) \in f[Y]\}$ and $M(x) = \max\{y \in [0,1] \mid (x,y) \in f[Y]\}$.

Clearly $m(x) \leq M(x)$, for all $x \in X$, and m, M are l.s.c, u.s.c., respectively, since $f[Y]$ is closed. Then (m, M) is a fancy pair of functions on X and $D_m^M = f[Y]$.

For (5) $\Rightarrow$ (4), suppose that there are a closed, non-empty, subset X of $2^{\mathbb{N}}$ and a fancy pair (m, M) of functions on X such that there is a homeomorphism $f : Y \to D_m^M$. Then f is the required injection.

It thus remains to establish (1) $\Rightarrow$ (4). By [Kur68, §46, V, Theorem 3], there is a continuous map $f_0 : Y \to 2^{\mathbb{N}}$ such that $f_0(x) = f_0(x')$ if and only if x, x' belong to the same connected component.

By [Car68], any compact metrizable space with a closed partial order can be embedded continuously and order-preservingly in $[0,1]^{\mathbb{N}}$ with the product order defined by $x \leq^{[0,1]^{\mathbb{N}}} y$ if and only if for all $n \in \mathbb{N}$, $x(n) \leq y(n)$. Let $h : Y \to [0,1]^{\mathbb{N}}$ be such an embedding. Let $f_1 : Y \to [0,1]$ be defined by $f_1(x) = d(\mathbf{0}, h(x))$, where d is the product metric on $[0,1]^{\mathbb{N}}$ and $\mathbf{0} = (0, 0, \dots)$. Then f_1 is the composition of two continuous functions, so it is continuous, and its restriction to each connected component of Y is injective, since $d(\mathbf{0}, x) < d(\mathbf{0}, y)$ whenever x is less than y in the product order on $[0,1]^{\mathbb{N}}$.

Let $f : Y \to 2^{\mathbb{N}} \times [0,1]$ be defined by $f(x) = (f_0(x), f_1(x))$. Then f is the continuous embedding which we were seeking. $\square$

Note that if $\preceq$ is the closed order on Y used for embedding Y into the Cantor fence, the embedding f of (1) $\Rightarrow$ (4) in the preceding proof also embeds $\preceq$ in $\trianglelefteq$.

For later use, we say that an order relation on the fence Y is *strongly compatible* if it satisfies (2) of Theorem 3.3.2. For example, $\trianglelefteq$ is a strongly compatible order on the Cantor fence.

Notice that any smooth fence Y can be given a $\mathcal{L}_R$-quotient structure by letting $\leq^Y$ be a strongly compatible order on Y. When we want to stress that we are considering a smooth fence with a strongly compatible order as a compact metrizable $\mathcal{L}_R$-quotient, we write $(Y, \leq^Y)$. Since there are no immediate successors in a strongly compatible order, it follows that $(Y, \leq^Y)$ is an H-forest.

Remark 3.3.3. Condition (2) in Theorem 3.3.2 implies that the ternary relation T on a smooth fence Y, defined by $T(x, y, x')$ if and only if $x = y = x'$ or y belongs to the arc with endpoints x, x', is closed. We do not know if requiring that this relation is closed is equivalent or strictly weaker than the conditions in Theorem 3.3.2.

3.4 Smooth fences and $\mathcal{F}_0$

The goal of this section is to prove that $[\![\mathcal{F}_0]\!]$ coincides with the class of smooth fences with strongly compatible orders. One direction is Theorem 3.4.1, the other Theorem 3.4.3.

Theorem 3.4.1. *Let (P_n, φ_n^m) be a fine projective sequence in $\mathcal{F}_0$, with projective limit $\mathbb{P}$ and let $p : \mathbb{P} \to \mathbb{P}/R^{\mathbb{P}}$ be the quotient map. Then $\mathbb{P}/R^{\mathbb{P}}$ is a smooth fence.*

The connected components of $\mathbb{P}/R^{\mathbb{P}}$ are the maximal chains of the order $\leq^{\mathbb{P}/R^{\mathbb{P}}}$. They are the sets of the form $p[B]$, where B is a maximal chain in $\mathbb{P}$; in particular, if B has more than two elements, then $p[B]$ is an arc.

Proof. The relation $\leq^{\mathbb{P}/R^{\mathbb{P}}}$ on $\mathbb{P}/R^{\mathbb{P}}$ is a closed order by Corollary 3.2.5.

If $x \not\leq^{\mathbb{P}/R^{\mathbb{P}}} y \not\leq^{\mathbb{P}/R^{\mathbb{P}}} x$, pick $u \in p^{-1}(x), v \in p^{-1}(y)$ and let $n \in \mathbb{N}$ be such that $\varphi_n(u) \not\leq \varphi_n(v) \not\leq \varphi_n(u)$. This implies that $\varphi_n(u), \varphi_n(v)$ belong to distinct maximal chains B, B', respectively, of P_n. Then $\varphi_n^{-1}(B), \varphi_n^{-1}(B')$ are clopen, $R^{\mathbb{P}}$-invariant subsets of $\mathbb{P}$ and, in turn, $p[\varphi_n^{-1}(B)], p[\varphi_n^{-1}(B')]$ are clopen subsets of $\mathbb{P}/R^{\mathbb{P}}$ separating x and y, so x, y belong to distinct connected components of $\mathbb{P}/R^{\mathbb{P}}$.

If $x \leq^{\mathbb{P}/R^{\mathbb{P}}} y$, let $u, v \in \mathbb{P}$ with $u \in p^{-1}(x), v \in p^{-1}(y), u \leq v$. Since $[u, v]$ is R-connected by Lemma 3.2.1, from Lemma 1.2.5 it follows that $p[[u, v]]$ is a connected subset of $\mathbb{P}/R^{\mathbb{P}}$ containing x, y. Therefore x, y belong to the same connected component.

These two facts show that the connected components of $\mathbb{P}/R^{\mathbb{P}}$ are the maximal chains of $\leq^{\mathbb{P}/R^{\mathbb{P}}}$ or, equivalently, the sets of the form $p[B]$, where B ranges over the maximal chains of $\mathbb{P}$. If in particular B has more than two points, then $p[B]$ is not a singleton by Lemma 3.2.2.

Thus it remains to show that the non-singleton connected components of $\mathbb{P}/R^{\mathbb{P}}$ are arcs. So let K be a non-singleton connected component of $\mathbb{P}/R^{\mathbb{P}}$. By the above, the restriction of $\leq^{\mathbb{P}/R^{\mathbb{P}}}$ to K is a closed total order, so it is complete as an order by Lemma 2.3.7, and has a minimum and a maximum that are distinct. Moreover, it is dense as K is connected, so it is a separable order as open intervals are open subsets in the topology of K. Using [Ros82, Theorem 2.30], the restriction of $\leq^{\mathbb{P}/R^{\mathbb{P}}}$ to K is an order of type $1 + \lambda + 1$, where λ is the order type of $\mathbb{R}$; as the sets of the form $\{x \in K \mid x <^{\mathbb{P}/R^{\mathbb{P}}} z\}$ and $\{x \in K \mid z <^{\mathbb{P}/R^{\mathbb{P}}} x\}$ are open subsets of K, this means that there is a continuous bijection $K \to [0, 1]$, which is therefore a homeomorphism. $\square$

The converse of Theorem 3.4.1 is proved in Corollary 3.4.4, for which we need the following lemma and definition.

Lemma 3.4.2. *Let X be a zero-dimensional compact metrizable space and (m, M) a fancy pair of functions on X. For each $\varepsilon > 0$ and each clopen partition $\mathcal{U}$ of X there is a clopen partition $\mathcal{W}$ refining $\mathcal{U}$, such that for all $U \in \mathcal{W}$ there is $x_U \in U$ such that:*

$$m(x_U) - \min\{m(x) \mid x \in U\} < \varepsilon, \qquad \max\{M(x) \mid x \in U\} - M(x_U) < \varepsilon. \tag{3.1}$$

Proof. By dealing with one element of $\mathcal{U}$ at a time, it is enough to show that given a zero-dimensional compact metrizable space X, a fancy pair (m, M), and $\varepsilon > 0$, there is a clopen partition $\mathcal{W} = \{W_0, \dots, W_k\}$ of X such that for all $U \in \mathcal{W}$ there is $x_U \in U$ for which (3.1) holds.

For any clopen set $U \subseteq X$, let

$$m_U = \min\{m(x) \mid x \in U\}, \qquad M_U = \max\{M(x) \mid x \in U\}.$$

If there exists $x_X \in X$ satisfying (3.1), then we are done by letting $k = 0, W_0 = X$. Otherwise, let $U_0 = \{x \in X \mid M(x) < M_X - \frac{\varepsilon}{2}\}$. This is an open set, and since there is no x_X satisfying (3.1), it contains the closed, non-empty, set $C_0 = \{x \in X \mid m(x) \leq m_X + \frac{\varepsilon}{2}\}$. By the zero-dimensionality of X and the compactness of C_0, let V_0 be clopen such that $C_0 \subseteq V_0 \subseteq U_0$. Notice that

$$m_{V_0} = m_X, \qquad M_{V_0} < M_X - \frac{\varepsilon}{2}.$$

If there exists $x_{V_0} \in V_0$ such that (3.1) holds, then set $W_0 = V_0$. Otherwise repeat the process within V_0, to find a clopen set V_1 with $C_0 \subseteq V_1 \subseteq V_0$ and

$$m_{V_1} = m_{V_0} = m_X, \qquad M_{V_1} < M_{V_0} - \frac{\varepsilon}{2} < M_X - \varepsilon.$$

Thus this process must stop, yielding finally a clopen subset W_0 such that $C_0 \subseteq W_0 \subseteq U_0$ and there exists $x_{W_0} \in W_0$ for which (3.1) holds.

Now start the process over again within $X' = X \setminus W_0$, which is non-empty by case assumption. Since $C_0 \subseteq W_0 \subseteq U_0$, it follows that

$$m_X + \frac{\varepsilon}{2} < m_{X'}, \qquad M_{X'} = M_X.$$

If there exists $x_{X'} \in X'$ satisfying (3.1), we are done by letting $k = 1, W_1 = X'$. Otherwise we eventually produce a clopen subset W_1 of X' containing $C_1 = \{x \in X' \mid m(x) \leq m_{X'} + \frac{\varepsilon}{2}\}$, contained in $U_1 = \{x \in X' \mid M(x) < M_{X'} - \frac{\varepsilon}{2}\}$, and such that there exists $x_{W_1} \in W_1$ satisfying (3.1). Set $X'' = X \setminus (W_0 \cup W_1)$ and notice that

$$m_X + \varepsilon < m_{X'} + \frac{\varepsilon}{2} < m_{X''}, \qquad M_{X''} = M_X.$$

Thus the process eventually stops, providing the desired partition $\mathcal{W}$. $\square$

Theorem 3.4.3. *Let $(Y, \leq^Y)$ be a smooth fence with a strongly compatible order. Then there is a $\mathcal{F}_0$-suitable sequence of regular quasi partitions of $(Y, \leq^Y)$.*

Together with Theorem 3.4.1 it follows that $[\![\mathcal{F}_0]\!]$ coincides with the class of smooth fences (with strongly compatible orders). By Proposition 1.4.3, we have the following corollary.

Corollary 3.4.4. *Let $(Y, \leq^Y)$ be a smooth fence with a strongly compatible order. There exists a fine projective sequence of structures (P_n, φ_n^m) from $\mathcal{F}_0$ closely approximating $(Y, \leq^Y)$ in such a way that for each $n \in \mathbb{N}$, $a, a' \in P_n$, it holds that $a \leq^{P_n} a'$ if and only if there are $x \in \operatorname{int}([\![a]\!]_{\varphi_n}), x' \in \operatorname{int}([\![a']\!]_{\varphi_n})$, $x \leq^Y x'$.*

Proof of Theorem 3.4.3. By Theorem 3.3.2 and the remark following it, we can assume that $Y = D_m^M$ for a closed, non-empty $X \subseteq 2^{\mathbb{N}}$ and a fancy pair (m, M) of functions on X, such that $\leq^Y$ coincides with the product order $\trianglelefteq$ on $X \times [0,1]$. We can furthermore assume that $m(x) > 0, M(x) < 1$ for all $x \in X$. Let d be the product metric on $X \times [0,1]$.

We first define a homeomorphic copy $Y' = D_{m'}^{M'}$ of Y in $X \times (0,1)$ and a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of partition of X such that for any $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$, there is $x_U \in U$ such that:

$$m'(x_U) = \min\{m'(x) \mid x \in U\}, \qquad M'(x_U) = \max\{M'(x) \mid x \in U\}.$$

This allows us to find a sequence of regular quasi-partitions of Y' which in turn give rise to the P_n's.

Let $\mathcal{U}_0 = \{X\}$ be the trivial clopen partition of X and $\beta_0 : X \times [0,1] \to X \times [0,1]$ be the identity. Suppose one has defined a clopen partition $\mathcal{U}_n$ of X and a homeomorphism $\beta_n : X \times [0,1] \to X \times [0,1]$. Let m^n, M^n be such that $D_{m^n}^{M^n} = \beta_n[Y]$. For any clopen set $U \subseteq X$, denote

$$m_U^n = \min_{x\in U} m^n(x), \qquad M_U^n = \max_{x\in U} M^n(x).$$

Let $\mathcal{U}_{n+1}$ refine $\mathcal{U}_n$, have mesh less than $\frac{1}{n+1}$, and satisfy Lemma 3.4.2 for $\beta_n[Y]$ and $\varepsilon = 1/2^{n+1}$. For each $U \in \mathcal{U}_{n+1}$ fix x_U given by Lemma 3.4.2, additionally we can ask that if $m_U^n < M_U^n$, then $m^n(x_U) < M^n(x_U)$.

For any $\ell \in \mathbb{N}$ and any two increasing sequences of real numbers $0 < a_0 < \cdots < a_{\ell-1} < 1$ and $0 < b_0 < \cdots < b_{\ell-1} < 1$, let $P_{\vec{a}}^{\vec{b}} : [0,1] \to [0,1]$ be the piecewise linear function mapping $0 \mapsto 0, 1 \mapsto 1, a_i \mapsto b_i$ for each $i < \ell$:

$$P_{\vec{a}}^{\vec{b}}(y) = \begin{cases} \dfrac{b_0}{a_0} y & \text{if } y \leq a_0, \\ \dfrac{b_{i+1} - b_i}{a_{i+1} - a_i} y + \dfrac{b_i a_{i+1} - a_i b_{i+1}}{a_{i+1} - a_i} & \text{if } a_i < y \leq a_{i+1}, i < \ell - 1, \\ \dfrac{1 - b_{\ell-1}}{1 - a_{\ell-1}} y + \dfrac{b_{\ell-1} - a_{\ell-1}}{1 - a_{\ell-1}} & \text{if } y > a_{\ell-1}. \end{cases}$$

Note that, for fixed ℓ, this is a continuous function of the variables $a_0, \ldots, a_{\ell-1}, y$.

If for each $x \in U$, $m^n(x) = M^n(x)$, then $m^n{\restriction_U} = M^n{\restriction_U} : U \to [0,1]$ is a continuous function, as it is both l.s.c. and u.s.c.. If follows that if we fix $x_U \in U$ and define $\alpha_U : U \times [0,1] \to U \times [0,1]$ as $\alpha_U(x,y) = \left(x, P_{m^n(x)}^{m^n(x_U)}(y)\right)$, then α_U is a homeomorphism. Notice that, in this case, α_U sends $\beta_n[Y] \cap (U \times [0,1])$ onto $U \times \{m^n(x_U)\}$; in particular, if $\beta_n[Y] \cap (U \times [0,1]) = U \times \{m^n(x_U)\}$, then α_U is the identity.

If, on the other hand, $x_U \in U$ is such that $m^n(x_U) < M^n(x_U)$, we define the functions $f_U, g_U, f'_U, g'_U : U \to (0,1)$ as follows.

$$f_U(x) = \begin{cases} m^n_U & \text{if } x \neq x_U \\ m^n(x_U) & \text{if } x = x_U \end{cases}$$
$$g_U(x) = \min\{m^n(x), m^n(x_U)\}$$
$$f'_U(x) = \begin{cases} M^n_U & \text{if } x \neq x_U \\ M^n(x_U) & \text{if } x = x_U \end{cases}$$
$$g'_U(x) = \max\{M^n(x), M^n(x_U)\}$$

It is immediate by their definitions that f_U, g'_U are u.s.c., g_U, f'_U are l.s.c., and that:

$$m^n_U \leq f_U \leq g_U \leq m^n(x_U) < M^n(x_U) \leq g'_U \leq f'_U \leq M^n_U.$$

By the Katětov–Tong insertion theorem there are $h_U, h'_U : U \to (0,1)$ continuous, such that $f_U \leq h_U \leq g_U$ and $g'_U \leq h'_U \leq f'_U$.

We define $\alpha_U : U \times [0,1] \to U \times [0,1]$ to be:

$$\alpha_U(x,y) = \left(x, P^{m^n_U, M^n_U}_{h_U(x), h'_U(x)}(y)\right).$$

Then α_U is a homeomorphism.

Define $\alpha_n = \bigsqcup_{U \in \mathcal{U}_{n+1}} \alpha_U$, so $\alpha_n \in \mathrm{Homeo}(X \times [0,1])$. Finally let $\beta_{n+1} = \alpha_n \beta_n$ and m^{n+1}, M^{n+1} be such that $\beta_{n+1}[Y] = D^{M^{n+1}}_{m^{n+1}}$. Notice that for any $U \in \mathcal{U}_{n+1}$

$$m^{n+1}(x_U) = m^n_U = m^{n+1}_U \quad \text{and} \quad M^{n+1}(x_U) = M^n_U = M^{n+1}_U. \tag{3.2}$$

Let $(x,y), (x,y') \in \beta_n[Y]$, and suppose that $x \in U \in \mathcal{U}_{n+1}$, $y \leq y'$. Then $m^n_U \leq h_U(x) \leq y \leq y' \leq h'_U(x) \leq M^n_U$ so:

$$P^{m^n_U, M^n_U}_{h_U(x), h'_U(x)}(y') - P^{m^n_U, M^n_U}_{h_U(x), h'_U(x)}(y) = \frac{M^n_U - m^n_U}{h'_U(x) - h_U(x)}(y' - y) \geq y' - y,$$

that is, $d((x,y),(x,y')) \leq d(\alpha_U(x,y), \alpha_U(x,y'))$. It follows that for $(x,y),(x,y') \in Y$:

$$d((x,y),(x,y')) \leq d(\beta_{n+1}(x,y), \beta_{n+1}(x,y')). \tag{3.3}$$

We prove that the sequence $(\beta_n)_{n\in\mathbb{N}}$ is Cauchy with respect to the supremum metric $d_{\sup}$. Indeed, for each n, $d_{\sup}(\mathrm{id}, \alpha_n) < 1/2^{n+1}$ by the definition of the points x_U. By right invariance of the supremum metric and the triangle inequality, whenever $n < m$,

$$d_{\sup}(\beta_n, \beta_m) = d_{\sup}(\beta_n, \alpha_{m-1} \cdots \alpha_n \beta_n) = d_{\sup}(\mathrm{id}, \alpha_{m-1} \cdots \alpha_n) \leq$$
$$\leq d_{\sup}(\mathrm{id}, \alpha_{m-1}) + \cdots + d_{\sup}(\mathrm{id}, \alpha_n) < \sum_{i=n+1}^{m} 1/2^i < 1/2^n.$$

It follows that for each ε, there is n such that for each $m > n$, $d_{\sup}(\beta_n, \beta_m) < \varepsilon$.

Since the space of continuous functions from $X \times [0,1]$ in itself with the supremum metric is complete, the sequence $(\beta_n)_{n\in\mathbb{N}}$ has a limit, which we denote by β. Since it is the limit of surjective functions, β is surjective. We prove that it is injective on Y, that is, that its restriction to Y is a homeomorphism onto $Y' = \beta[Y]$.

Let $(x,y),(x',y') \in Y$. If $x \neq x'$, then $\beta(x,y) \neq \beta(x',y')$ as β is the identity on the first coordinate. So suppose $x = x'$. Since (3.3) holds for each $n \in \mathbb{N}$, we have that $d((x,y),(x,y')) \leq d(\beta(x,y),\beta(x,y'))$, so β is injective on Y.

By (3.2) it follows that $Y' \subseteq X \times [m_X, M_X] \subseteq X \times (0,1)$. Notice that $x \trianglelefteq x'$ if and only if $\beta(x) \trianglelefteq \beta(x')$. Let m', M' be such that $D_{m'}^{M'} = Y'$. For any $n \in \mathbb{N}$ and $U \in \mathcal{U}_{n+1}$, $m'(x_U) = m'_U$ and $M'(x_U) = M'_U$. This is clear if $m^n{\restriction_U} = M^n{\restriction_U}$. Otherwise, we have seen that $m^{n+1}(x_U) = m_U^{n+1}$. Assume that $m^r(x_U) = m_U^r$ for some $r \geq n+1$. Given any $U' \in \mathcal{U}_{r+1}$ with $U' \subseteq U$, by (3.2) it follows that $m^r(x_U) \leq m_{U'}^r = m_{U'}^{r+1}$, whence $m^r(x_U) = m_U^r = m_U^{r+1}$ and, in particular, $\forall r \geq n+1$ $m^r(x_U) = m^{n+1}(x_U) = m_U^r$, which allows to conclude $m'(x_U) = m^{n+1}(x_U) = m'_U$. Similarly, $M'(x_U) = M'_U$.

Let $K_U = \{(x_U, y) \mid m'_U \leq y \leq M'_U\} = (\{x_U\} \times [0,1]) \cap Y'$.

Let $x_0 = 0, x_1 = 1$. Let $\Theta = \left\{x_{m/2^n} \,\middle|\, n \geq 1, 1 \leq m < 2^n\right\}$ be a countable dense subset of $(0,1) \setminus \{m_U, M_U \mid U \in \mathcal{U}_n, n \in \mathbb{N}\}$, indexed in such a way that $x_p < x_q$ if and only if $p < q$.

For $n \geq 0$, let:

$$\mathcal{I}_n = \left\{\left[x_{m/2^n}, x_{(m+1)/2^n}\right] \,\middle|\, 0 \leq m \leq 2^n - 1\right\}.$$

Then define:

$$\mathcal{C}_n = \{U \times I \mid U \in \mathcal{U}_n, I \in \mathcal{I}_n\}.$$

Notice that for each n:

1. $\mathcal{C}_n$ is a regular quasi-partition of $X \times [0,1]$;
2. $\forall C \in \mathcal{C}_{n+1}\ \exists! C' \in \mathcal{C}_n\ C \subseteq C'$;
3. The mesh of $\mathcal{C}_n$ tends to 0 as n grows, since Θ is dense and the mesh of $\mathcal{U}_n$ goes to 0.

Endow each $\mathcal{C}_n$ with the discrete topology and give $\mathcal{C}_n$ an $\mathcal{L}_R$-structure by letting

- $C\, R^{\mathcal{C}_n}\, C'$ if and only if $C \cap C' \neq \emptyset$,
- $C \leq^{\mathcal{C}_n} C'$ if and only if there are $x \in \mathrm{int}(C), x' \in \mathrm{int}(C')$ with $x \trianglelefteq x'$.

Then $\mathcal{C}_n \in \mathcal{F}_0$. Notice that C, C' are $\leq^{\mathcal{C}_n}$-comparable if and only if $\pi_1[C] = \pi_1[C']$, where π_1 is the projection onto X.

For each n, define $P_n = \{C \in \mathcal{C}_n \mid C \cap Y' \neq \emptyset\}$, and have it inherit the $\mathcal{L}_R$-structure of $\mathcal{C}_n$. Let $\mathcal{D}_n = \left\{\beta^{-1}(C \cap Y') \,\middle|\, C \in P_n\right\}$. We prove that $P_n \in \mathcal{F}_0$, $\mathcal{D}_n$ is P_n-like, and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ is a $\mathcal{F}_0$-suitable sequence of $(Y, \leq^Y)$.

Claim 3.4.4.1. For each $C \in P_n$, $\mathrm{int}(C) \cap K_{\pi_1[C]} \neq \emptyset$.

Proof. Let $(x, y) \in C \cap Y'$. As $m'(x_{\pi_1[C]}) = m'_{\pi_1[C]}$ and $M'(x_{\pi_1[C]}) = M'_{\pi_1[C]}$, it follows that $(x_{\pi_1[C]}, y) \in C \cap K_{\pi_1[C]}$. The projections of endpoints of $K_{\pi_1[C]}$ on the second coordinate do not belong to Θ, so $\mathrm{int}(C) \cap K_{\pi_1[C]} \neq \emptyset$. □

Claim 3.4.4.2. $\{C \cap Y' \mid C \in P_n\}$ is a regular quasi-partition of Y'.

Proof. We first show that P_n is a covering of Y': given $(x, y) \in Y'$, let $U \in \mathcal{U}_n$ be such that $x \in U$; then $m'(x_U) \leq y \leq M'(x_U)$, so that $(x_U, y) \in K_U$. If $I \in \mathcal{I}_n$ is such that $y \in I$, then $(x, y) \in U \times I \in P_n$.

We show that for each $C \in P_n$, $C \cap Y'$ is regular in Y', that is, that

$$\mathrm{cl}_{X\times[0,1]}(\mathrm{int}_{X\times[0,1]}(C) \cap Y') = C \cap Y'.$$

Let $(x, y) \in C \cap Y'$ and O be an open neighborhood of (x, y) in $X \times [0, 1]$. There exist $m > n$ and $C' \in P_m$ such that $y \in C' \subseteq O$ and $C' \cap C$. By Claim 3.4.4.1 $\mathrm{int}(C') \cap K_{\pi_1[C']} \neq \emptyset$, so $O \cap \mathrm{int}(C) \cap Y \neq \emptyset$, and we are done.

If $C, C' \in P_n$ are distinct, by item 1 above, $C \cap C' \cap Y' \subseteq \partial(C) \cap \partial(C') \cap Y' \subseteq \partial_{Y'}(C \cap Y') \cap \partial_{Y'}(C' \cap Y')$, so it is a quasi-partition. □

Claim 3.4.4.3. $\{C \cap Y' \mid C \in P_n\}$ is P_n-like.

Proof. (A0) holds by Claim 3.4.4.2. If $C, C' \in P_n$ are such that $C\ R^{P_n}\ C'$, then $C\ R^{\mathcal{C}_n}\ C'$, which in particular entails that $\pi_1[C] = \pi_1[C']$. So $C \cap C' \cap K_{\pi_1[C]} \neq \emptyset$ and thus $C \cap C' \cap Y' \neq \emptyset$. If $C \cap C' \cap Y' \neq \emptyset$ then $C \cap C' \neq \emptyset$ so $C\ R^{\mathcal{C}_n}\ C'$ and finally $C\ R^{P_n}\ C'$. Therefore (A1) holds.

Let $(x, y) \leq^{Y'} (x, y')$. If $C, C' \in P_n$ are such that $(x, y) \in C, (x, y') \in C'$, then $C \leq^{\mathcal{C}_n} C'$ by definition of $\leq^{\mathcal{C}_n}$ and therefore $C \leq^{P_n} C'$, which takes care of (A2).

Let $C, C' \in P_n$, be such that $C \leq^{P_n} C'$. Then $\pi_1[C] = \pi_1[C']$, and by Claim 3.4.4.1 there are $(x, y) \in \mathrm{int}(C) \cap K_{\pi_1[C]}, (x, y') \in \mathrm{int}(C') \cap K_{\pi_1[C]}$, with $(x, y) \leq^{Y'} (x, y')$. Thus (A3) holds. □

Claim 3.4.4.4. $P_n \in \mathcal{F}_0$

Proof. Suppose $C, C' \in P_n$ and $D \in \mathcal{C}_n$ with $C \leq^{\mathcal{C}_n} D \leq^{\mathcal{C}_n} C'$. Then $K_{\pi_1[D]} \cap D \neq \emptyset$, so $D \in P_n$. Therefore P_n is a $\leq^{\mathcal{C}_n}$-convex substructure of $\mathcal{C}_n$, so $P_n \in \mathcal{F}_0$. □

By Lemma 1.2.7, $\mathcal{D}_n$ is P_n-like, since $\beta_{\restriction Y}$ is an isomorphism. From Items 2 and 3 above and the fact that $\beta_{\restriction Y}$ is an isomorphism, it follows that $\mathcal{D}_{n+1}$ refines $\mathcal{D}_n$ and that the mesh goes to 0, which concludes the proof.

□

As mentioned in the introduction, in [BK15] the Lelek fan is obtained as a quotient of the projective Fraïssé limit of a subclass of $\mathcal{F}$. In particular, the Lelek fan is approximable by a fine projective sequence from $\mathcal{F}$. We therefore raise the following question, an answer to which would involve proving analogs of Theorems 3.4.1 and 3.4.3 for $\mathcal{F}$.

Question 3.4.5. What is the class of spaces which are approximable by fine projective sequences from $\mathcal{F}$?

3.5 Spaces of endpoints of smooth fences

Given a smooth fence $(Y, \leq^Y)$ with a strongly compatible order, let $\mathfrak{L}(Y, \leq^Y)$, $\mathfrak{U}(Y, \leq^Y)$ be the space of $\leq^Y$-minimal points of Y and the space of $\leq^Y$-maximal points of Y, respectively. By the definition of a strongly compatible order, in these sets are contained all endpoints of Y:

$$\mathrm{E}(Y) = \mathfrak{L}(Y, \leq^Y) \cup \mathfrak{U}(Y, \leq^Y).$$

Notice that $x \in \mathfrak{L}(Y, \leq^Y) \cap \mathfrak{U}(Y, \leq^Y)$ if and only if $\{x\}$ is a connected component of Y. When the order $\leq^Y$ is clear from context we suppress the mention of it in $\mathfrak{L}(Y, \leq^Y)$ and $\mathfrak{U}(Y, \leq^Y)$.

Remark 3.5.1. By Theorem 3.3.2, Y is homeomorphic to D_m^M for some fancy pair (m, M) of functions with domain a closed subset of $2^{\mathbb{N}}$. It follows that $\mathfrak{L}(Y, \leq^Y)$, $\mathfrak{U}(Y, \leq^Y)$ are homeomorphic to the graphs of m, M, respectively.

In this subsection we establish some topological properties of spaces of endpoints of smooth fences. In particular, we concentrate on the spaces $\mathfrak{L}(Y, \leq^Y)$, $\mathfrak{U}(Y, \leq^Y)$, $\mathfrak{L}(Y, \leq^Y) \cap \mathfrak{U}(Y, \leq^Y)$. We therefore fix a smooth fence $(Y, \leq^Y)$ with a strongly compatible order. By Corollary 3.4.4 we can assume that $Y = \mathbb{P}/R^{\mathbb{P}}$ for some fine projective sequence (P_n, φ_n^m) in $\mathcal{F}_0$ with projective limit $\mathbb{P}$, and that $\leq^Y$ is $\leq^{\mathbb{P}/R^{\mathbb{P}}}$. Let $p : \mathbb{P} \to \mathbb{P}/R^{\mathbb{P}}$ be the quotient map.

Lemma 3.5.2. *A point $u \in \mathbb{P}$ is $\leq^{\mathbb{P}}$-maximal if and only if for any $n \in \mathbb{N}$ there exists $m > n$ such that $\varphi_n^m(\max\{a \in P_m \mid \varphi_m(u) \leq a\}) = \varphi_n(u)$, and $u \in \mathbb{P}$ is $\leq^{\mathbb{P}}$-minimal if and only if for any $n \in \mathbb{N}$ there exists $m > n$ such that $\varphi_n^m(\min\{a \in P_m \mid a \leq \varphi_m(u)\}) = \varphi_n(u)$.*

Proof. Suppose u is $\leq^{\mathbb{P}}$-maximal and fix $n \in \mathbb{N}$. For $m > n$, let $b_m = \max\{a \in P_m \mid \varphi_m(u) \leq a\}$. If for every $m > n$ it holds that $\varphi_n^m(b_m) > \varphi_n(u)$, let $v_m \in \varphi_m^{-1}(b_m), u_m \in \varphi_m^{-1}(\varphi_m(u))$ be such that $u_m \leq v_m$. A subsequence v_{m_k} converges to some v. It follows that $u \leq^{\mathbb{P}} v$, as $u = \lim_{m\to\infty} u_m$ and the order is closed, and $u \neq v$ as $\varphi_n(v_m) \neq \varphi_n(u)$, for any $m > n$, a contradiction with the maximality of u.

Conversely, let $u \in \mathbb{P}$ be such that for each $n \in \mathbb{N}$ there exists $m > n$ such that $\varphi_n^m(\max\{a \in P_m \mid \varphi_m(u) \leq a\}) = \varphi_n(u)$ and let $u \leq^{\mathbb{P}} v$. Fix n, with the objective of showing $\varphi_n(u) = \varphi_n(v)$. Let $m > n$ satisfy the hypothesis; notice that it implies that $\varphi_n^m[\{a \in P_m \mid \varphi_m(u) \le a\}] = \{\varphi_n(u)\}$. From $u \le v$ it follows that $\varphi_m(u) \le \varphi_m(v)$ so $\varphi_n(v) = \varphi_n^m \varphi_m(v) = \varphi_n(u)$.

The case of u $\le^{\mathbb{P}}$-minimal is symmetrical. $\square$

Corollary 3.5.3. *Given $x \in \mathfrak{U}(Y)$ and any open neighborhood O of x in Y, for m big enough the following holds: if $B_m \in \mathrm{MC}(P_m)$ is such that $x \in \bigcup_{a \in B_m} [\![a]\!]_{\varphi_m}$, then $[\![\max B_m]\!]_{\varphi_m} \subseteq O$. Consequently, $\lim_{m\to\infty} [\![\max B_m]\!]_{\varphi_m} = \{x\}$.*

The same holds for $x \in \mathfrak{L}(Y)$, upon changing max *to* min.

Proof. Let $u = \max p^{-1}(x)$ and $n \in \mathbb{N}$ be such that $[\![\varphi_n(u)]\!]_{\varphi_n} \subseteq O$. By Lemma 3.5.2 there is $m > n$ such that $\varphi_n^m(\max B_m) = \varphi_n(u)$, for $B_m \in \mathrm{MC}(P_m)$ with $\varphi_m(u) \in B_m$. This implies that for all $m' \ge m$ if $B_{m'} \in \mathrm{MC}(P_{m'})$ is such that $\varphi_{m'}(u) \in B_{m'}$ then $\varphi_n^{m'}(\max B_{m'}) = \varphi_n(u)$. It follows that eventually $[\![\max B_m]\!]_{\varphi_m} \subseteq [\![\varphi_n(u)]\!]_{\varphi_n} \subseteq O$. $\square$

Corollary 3.5.4. *For any connected component $K \subseteq Y$ and any open neighborhood O of K in Y, there are $m \in \mathbb{N}, B \in \mathrm{MC}(P_m)$ such that $K \subseteq \bigcup_{a \in B} [\![a]\!]_{\varphi_m} \subseteq O$.*

Proof. It can be assumed that $O \ne Y$. Fix a compatible metric on Y and let δ be the distance between K and $Y \backslash O$. Let $u = \min p^{-1}(K), v = \max p^{-1}(K)$ and $n \in \mathbb{N}$ be such that the mesh of $[\![P_n]\!]_{\varphi_n}$ is less than δ, so that if $a \in P_n$ is such that $[\![a]\!]_{\varphi_n} \cap K \ne \emptyset$, then $[\![a]\!]_{\varphi_n} \subseteq O$. By Lemma 3.5.2 there are $m' > n$ and $B' \in \mathrm{MC}(P_{m'})$ with $\varphi_{m'}(u) \in B'$ and $\varphi_n^{m'}(\min B') = \varphi_n(u)$. By a second application of Lemma 3.5.2, there are $m > m', B \in \mathrm{MC}(P_m)$ such that $\varphi_m(v) \in B, \varphi_{m'}^m(\max B) = \varphi_{m'}(v)$, so $\varphi_n^m(\max B) = \varphi_n(v)$. Since $\varphi_{m'}^m(\min B) \ge \min B'$, it follows that $\varphi_n^m(\min B) \ge \varphi_n^{m'}(\min B') = \varphi_n(u)$ by virtue of $\varphi_n^{m'}$ being an epimorphism. If $a \in B$, then $\varphi_n(u) \le \varphi_n^m(a) \le \varphi_n(v)$, so $[\![\varphi_n^m(a)]\!]_{\varphi_n} \cap K \ne \emptyset$, hence $[\![a]\!]_{\varphi_m} \subseteq [\![\varphi_n^m(a)]\!]_{\varphi_n} \subseteq O$. It follows that $\bigcup_{a \in B} [\![a]\!]_{\varphi_m} \subseteq O$. $\square$

Proposition 3.5.5. *Each point of $\mathfrak{L}(Y) \cap \mathfrak{U}(Y)$ has a basis of neighborhoods in Y consisting of clopen sets. In particular, the space $\mathfrak{L}(Y) \cap \mathfrak{U}(Y)$ is zero-dimensional.*

Proof. Let $x \in \mathfrak{L}(Y) \cap \mathfrak{U}(Y)$ and O be an open neighborhood of x in Y. By Corollary 3.5.4 there exist $n \in \mathbb{N}$ and $B \in \mathrm{MC}(P_n)$ such that $x \in \bigcup_{a \in B} [\![a]\!]_{\varphi_n} \subseteq O$. By Lemma 3.2.6, $\bigcup_{a \in B} [\![a]\!]_{\varphi_n}$ is clopen in Y and so its trace in $\mathfrak{L}(Y) \cap \mathfrak{U}(Y)$ is clopen in $\mathfrak{L}(Y) \cap \mathfrak{U}(Y)$. $\square$

Since $\mathfrak{L}(Y)$ and $\mathfrak{U}(Y)$ are homeomorphic to graphs of semi-continuous functions with a zero-dimensional domain, by [DvM10, Remark 4.2] we have the following:

Proposition 3.5.6. *The spaces $\mathfrak{L}(Y)$ and $\mathfrak{U}(Y)$ are almost zero-dimensional.*

Lemma 3.5.7. *The spaces $\mathfrak{L}(Y), \mathfrak{U}(Y)$ are Polish.*

Proof. The set $\mathfrak{U}(Y) = \{x \in Y \mid \forall y \in Y, y \leq^Y x \vee (x \not\leq^Y y \wedge y \not\leq^Y x)\}$ is the co-projection of $\{(x,y) \mid y \leq^Y x \vee (x \not\leq^Y y \wedge y \not\leq^Y x)\}$, which is the union of a closed set and an open set of Y^2, since $\leq^Y$ is closed. A union of a closed set and an open set is G_δ and since Y is compact, the co-projection of an open set is open. Finally, as co-projection and intersection commute, the co-projection of a G_δ is G_δ. We conclude that $\mathfrak{U}(Y)$ is a G_δ subset of Y, thus is Polish.

Similarly for $\mathfrak{L}(Y)$. □

Corollary 3.5.8. *The spaces* $\mathrm{E}(Y)$ *and* $\mathfrak{L}(Y) \cap \mathfrak{U}(Y)$ *are Polish.*

Remark 3.5.9. The spaces $\mathfrak{L}(Y) \setminus \mathfrak{U}(Y)$ and $\mathfrak{U}(Y) \setminus \mathfrak{L}(Y)$ are strongly σ-complete spaces (that is, they are union of countably many closed and completely metrizable subspaces), since they are F_σ subsets of a Polish space.

Chapter 4

The Fraïssé fence

We denote by $\mathbb{F}$ the projective Fraïssé limit of $\mathcal{F}$. Recall from Corollary 3.1.9 that $\mathcal{F}_0$ is a projective Fraïssé family, with the same projective Fraïssé limit as $\mathcal{F}$. Therefore, for the remainder of this chapter we fix a fundamental sequence (F_n, γ_n^m) in $\mathcal{F}_0$, with F_0 consisting of a single element.

Proposition 4.0.1. $\mathbb{F}$ *is a close prespace.*

Proof. We show that the sequence (F_n, γ_n^m) is fine and the quotient map $p : \mathbb{F} \to \mathbb{F}/R^{\mathbb{F}}$ is irreducible. Let $a, b \in F_n$ have R^{F_n}-distance 2. Say, without loss of generality, $a\, R^{F_n}\, c\, R^{F_n}\, b$ and $a <^{F_n} c <^{F_n} b$. Consider $P \in \mathcal{F}_0$ obtained by F_n by blowing c up to two points. More precisely, let c_0, c_1 be two new elements, let $P = (F_n \setminus \{c\}) \cup \{c_0, c_1\}$, and define $\leq^P, R^P$ by extending the corresponding relations on $F_n \setminus \{c\}$ requiring $a <^P c_0 <^P c_1 <^P b, a\, R^P\, c_0\, R^P\, c_1\, R^P\, b$. Let $\varphi : P \to F_n$ be defined by:

$$\varphi(d) = \begin{cases} d & \text{if } d \in F_n, \\ c & \text{if } d \in \{c_0, c_1\}. \end{cases}$$

Then φ is an epimorphism by Lemma 3.1.8, and by (F2) there exist $m > n$ and an epimorphism $\theta : F_m \to P$ such that $\varphi\theta = \gamma_n^m$. Let $a' \in (\gamma_n^m)^{-1}(a), b' \in (\gamma_n^m)^{-1}(b)$, then $\theta(a') = a, \theta(b') = b$. If there was $c' \in F_m$ such that $a'\, R^{F_m}\, c'\, R^{F_m}\, b'$, then $\theta(c')$ should be R^P-connected to a and b, but no such element exists in P. By Lemma 1.2.3, (F_n, γ_n^m) is therefore fine.

To prove irreducibility of the quotient map, by Lemma 1.3.4 it suffices to show that for each $n \in \mathbb{N}$ and $a \in F_n$ there are $m > n$ and $b \in F_m$ such that $b'\, R^{F_m}\, b$ implies $\gamma_n^m(b') = a$. To this end fix n, a as above and define $P = F_n \sqcup \{a_0, a_1, a_2\}$ with $a_0\, R^P\, a_1\, R^P\, a_2$ and $a_0 <^P a_1 <^P a_2$, so that $\{a_0, a_1, a_2\} \in \mathrm{MC}(P)$ and $P \in \mathcal{F}_0$. Let $\varphi : P \to F_n$ be the identity restricted to F_n and $\varphi(a_i) = a$ for $0 \leq i \leq 2$. By Lemma 3.1.8, φ is an epimorphism and by (F2) there exist $m > n$ and an epimorphism $\theta : F_m \to P$ such that $\varphi\theta = \gamma_n^m$. Let $b \in \theta^{-1}(a_1)$ and $b'\, R^{F_m}\, b$, then $\theta(b') \in \{a_0, a_1, a_2\}$, so $\gamma_n^m(b) = a$. $\square$

4.1 A topological characterization

The study of the quotient $\mathbb{F}/R^{\mathbb{F}}$ is the main goal of this chapter. By Theorem 3.4.1, $\mathbb{F}/R^{\mathbb{F}}$ is a smooth fence. We call *Fraïssé fence* any space homeomorphic to $\mathbb{F}/R^{\mathbb{F}}$.

The following property of the Fraïssé fence is of crucial importance for its characterization. In particular it implies that $[\![F_n]\!]_{\gamma_n}$ is F_n-like, and thus that $([\![F_n]\!]_{\gamma_n})_{n\in\mathbb{N}}$ is a $\mathcal{F}_0$-suitable sequence of the Fraïssé fence.

Lemma 4.1.1. *Let $\varphi : \mathbb{F} \to P$ be an epimorphism onto some $P \in \mathcal{F}_0$. If $a, a' \in P$ with $a \leq a'$, there is an arc component of $\mathbb{F}/R^{\mathbb{F}}$ whose endpoints belong to $\operatorname{int}([\![a]\!]_\varphi)$, $\operatorname{int}([\![a']\!]_\varphi)$, respectively.*

Proof. Let $a_1, \ldots, a_\ell \in P$ be such that

$$a <^P a_1 <^P \ldots <^P a_\ell <^P a',$$
$$a\, R^P\, a_1 R^P \ldots R^P\, a_\ell\, R^P\, a'.$$

Notice that $\ell = 0$ if $a\, R^P\, a'$, in particular when $a = a'$.

Let $Q = P \sqcup \{b, c, d_1, \ldots, d_\ell, b', c'\} \in \mathcal{F}_0$, where

$$b <^Q c < d_1 <^Q \ldots <^Q d_\ell <^Q b' <^Q c',$$
$$b\, R^Q\, c\, R^Q\, d_1 R^Q \ldots R^Q\, d_\ell\, R^Q\, b'\, R^Q\, c',$$

such that $\{b, c, d_1, \ldots, d_\ell, b', c'\} \in \mathrm{MC}(Q)$. Let $\psi : Q \to P$ be the epimorphism defined as the identity on P and by letting

$$\begin{cases} \psi(b) = \psi(c) = a, \\ \psi(d_1) = a_1, \\ \ldots \\ \psi(d_\ell) = a_\ell, \\ \psi(b') = \psi(c') = a'. \end{cases}$$

By (L3′) there is an epimorphism $\theta : \mathbb{F} \to Q$ such that $\varphi = \psi\theta$. Let $u, u' \in \mathbb{F}$ with $\theta(u) = b, \theta(u') = c', u \leq^{\mathbb{F}} u'$. Let $v \in \mathbb{F}$ be minimal with $v \leq^{\mathbb{F}} u$. If $w\, R^{\mathbb{F}}\, v$, then $\theta(w)$ is either b or c, so $\varphi(w) = a$; similarly, if $v' \in \mathbb{F}$ is maximal with $u' \leq^{\mathbb{F}} v'$, and $w'\, R^{\mathbb{F}}\, v'$ then $\varphi(w') = a'$. So, by Lemma 1.3.8, $p(v) \in \operatorname{int}([\![a]\!]_\varphi), p(v') \in \operatorname{int}([\![a']\!]_\varphi)$. This implies that the connected component with endpoints $p(v), p(v')$ has the property we are looking for. □

The following theorem gives a topological characterization of the Fraïssé fence.

Theorem 4.1.2. *A smooth fence Y is a Fraïssé fence if and only if for any two open sets $O, O' \subseteq Y$ which meet a common connected component there is an arc component of Y whose endpoints belong to O, O', respectively.*

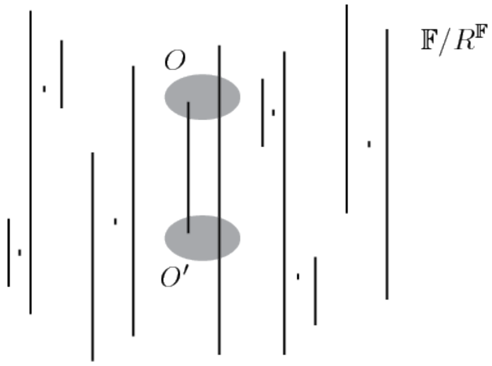

Figure 4.1: A representation of the characteristic property of the Fraïssé Fence.

The following lemmas are used in the proof of Theorem 4.1.2.

Lemma 4.1.3. *Let A, B, B' be HLOs and let $\varphi : B \to A$ and $\psi : B' \to A$ be $\mathcal{L}_R$-preserving maps such that $\psi[B'] \subseteq \varphi[B]$. Let $a_0 = \psi(\min B'), a_1 = \psi(\max B')$ and $r = \max\{|\varphi^{-1}(a)| \mid a \in A\}$. If $|\psi^{-1}(a)| \geq r$ for each $a \in \psi[B'] \setminus \{a_0, a_1\}$, then there exists an $\mathcal{L}_R$-preserving map $\theta : B' \to B$ such that $\varphi\theta = \psi$. Moreover:*

1. *if $\psi[B'] = \varphi[B]$ and $|\psi^{-1}(a_0)|, |\psi^{-1}(a_1)| \geq r$, then θ can be chosen to be surjective;*
2. *if $\psi[B'] = \varphi[B]$ and $|\varphi^{-1}(a_0)| = |\varphi^{-1}(a_1)| = 1$, then θ can be chosen to be surjective;*
3. *if $a \in A$, $b \in \varphi^{-1}(a), b' \in \psi^{-1}(a)$ and*
$$\min\{|\{c \in B' \mid \psi(c) = a, c < b'\}|, |\{c \in B' \mid \psi(c) = a, c > b'\}|\} \geq r - 1,$$
then θ can be chosen such that $\theta(b') = b$.

Proof. For each $a \in \psi[B'] \setminus \{a_0, a_1\}$ let θ map $\psi^{-1}(a)$ to $\varphi^{-1}(a)$ surjectively and monotonically. If $\psi[B'] = \varphi[B]$ and $|\psi^{-1}(a_0)|, |\psi^{-1}(a_1)| \geq r$, doing the same for $\psi^{-1}(a_0), \psi^{-1}(a_1)$ provides a map onto B. Otherwise, map all of $\psi^{-1}(a_0)$ to the maximal element of $\varphi^{-1}(a_0)$, and all of $\psi^{-1}(a_1)$ to the minimal element of $\varphi^{-1}(a_1)$. In the hypothesis of point (2), this produces a surjective map on B.

As for point (3), map $\{c \in B' \mid \psi(c) = a, c \leq b'\}, \{c \in B' \mid \psi(c) = a, c \geq b'\}$ monotonically onto $\{c \in B \mid \varphi(c) = a, c \leq b\}, \{c \in B \mid \varphi(c) = a, c \geq b\}$, respectively, so in particular $\theta(b') = b$. □

Lemma 4.1.4. *Let (P_n, φ_n^m) be a fine projective sequence in $\mathcal{F}_0$, with projective limit $\mathbb{P}$, and the quotient map $p : \mathbb{P} \to \mathbb{P}/R^{\mathbb{P}}$ be irreducible. Let $J^1, \ldots, J^\ell$ be connected components of $\mathbb{P}/R^{\mathbb{P}}$. For each $n \in \mathbb{N}$ and $1 \leq i \leq \ell$, let $J_n^i = \varphi_n[p^{-1}(J^i)]$ and $B_n^i \in \mathrm{MC}(P_n)$ be such that $J_n^i \subseteq B_n^i$. For any $n, r \in \mathbb{N}$, if the endpoints of the J^i's belong to $\bigcup_{a \in P_n} \mathrm{int}([\![a]\!]_{\varphi_n})$, there is $m_0 > n$ such that, for each $m \geq m_0$ and $1 \leq i \leq \ell$:*

(a) $\varphi_n^m[B_m^i] = J_n^i$,

(b) if J^i *is an arc, then* $\left|J_m^i \cap (\varphi_n^m)^{-1}(a)\right| > r$ *for each* $a \in J_n^i$.

Proof. We can suppose that the J^i's are distinct. Let $O_1, \dots, O_\ell$ be pairwise disjoint open neighborhoods of $J^1, \dots, J^\ell$, respectively, such that $O_i \subseteq \bigcup_{a \in J_n^i} [\![a]\!]_{\varphi_n}$, for $1 \leq i \leq \ell$. By Corollary 3.5.4, there is $m' > n$ such that for $1 \leq i \leq \ell$, one has $\bigcup_{a \in B_{m'}^i} [\![a]\!]_{\varphi_{m'}} \subseteq O_i$, that is, $\varphi_n^{m'}[B_{m'}^i] = J_n^i$. It follows that for all $m > m'$ and $1 \leq i \leq \ell$, one has $\varphi_n^m[B_m^i] = J_n^i$. For $1 \leq i \leq \ell$ such that J^i is an arc, and each $a \in J_n^i$, the set $[\![a]\!]_{\varphi_n} \cap J^i$ has more than one element; since the mesh of $[\![P_m]\!]_{\varphi_m}$ goes to 0, there exists $m_0 > m'$ such that for all $m > m_0$ condition (b) is satisfied. $\square$

Proof of Theorem 4.1.2. For the forward implication, it suffices to prove the conclusion for $\mathbb{F}/R^{\mathbb{F}}$. Let $O, O' \subseteq \mathbb{F}/R^{\mathbb{F}}$ be open sets which meet a common connected component K. Let $n \in \mathbb{N}, a, a' \in F_n$ be such that

$$[\![a]\!]_{\gamma_n} \subseteq O, \quad [\![a']\!]_{\gamma_n} \subseteq O', \quad \mathrm{int}([\![a]\!]_{\gamma_n}) \cap K \neq \emptyset \neq \mathrm{int}([\![a']\!]_{\gamma_n}) \cap K.$$

It follows that a, a' are $\leq^{F_n}$-comparable, so by Lemma 4.1.1 there is an arc component J of $\mathbb{F}/R^{\mathbb{F}}$ whose endpoints belong to $\mathrm{int}([\![a]\!]_{\gamma_n}), \mathrm{int}([\![a']\!]_{\gamma_n})$, respectively, and so to O, O', respectively.

Conversely, assume that for any open sets $O, O' \subseteq Y$ meeting a common connected component there is an arc component of Y whose endpoints belong to O, O', respectively. Let $\leq^Y$ be a strongly compatible order on Y. Let (P_n, φ_n^m) be the projective sequence given by Corollary 3.4.4, and let $\mathbb{Y}$ be its projective limit.

It is then enough to prove that $\mathbb{Y}$ is a projective Fraïssé limit of $\mathcal{F}_0$. To this end, by Proposition 1.1.2, we must prove that given $P \in \mathcal{F}_0$ and an epimorphism $\varphi : P \to P_n$, there are $m \geq n$ and an epimorphism $\psi : P_m \to P$ such that $\varphi\psi = \chi_n^m$. Let $r = \max\{|\varphi^{-1}(C)| \mid C \in P_n\}$ and $B^1, \dots, B^\ell$ be an enumeration of $\mathrm{MC}(P)$.

From $\min B^i \leq^P \max B^i$ it follows that $\varphi(\min B^i) \leq^{P_n} \varphi(\max B^i)$, for $1 \leq i \leq \ell$. There is a connected component of Y which meets the interior of both $[\![\varphi(\min B^i)]\!]_{\varphi_n}$, $[\![\varphi(\max B^i)]\!]_{\varphi_n}$, so by hypothesis there is an arc component J^i of Y whose endpoints belong to $\mathrm{int}[\![\varphi(\min B^i)]\!]_{\varphi_n}$, $\mathrm{int}[\![\varphi(\max B^i)]\!]_{\varphi_n}$, respectively. Notice that if $j \neq i$ is such that $\varphi[B^j] = \varphi[B^i]$, one can find a connected component J^j disjoint from J^i, by applying the hypothesis to a couple of open sets $O \subseteq [\![\varphi(\min B^i)]\!]_{\varphi_n}, O' \subseteq [\![\varphi(\max B^i)]\!]_{\varphi_n}$ which intersect J^i but avoid its endpoints.

By Lemma 4.1.4 there is $m_0 > n$ such that for all $m \geq m_0$ there are $A^1, \dots, A^\ell \in \mathrm{MC}(P_m)$ distinct such that one has $\varphi_n^m[A^i] = \varphi[B^i]$ and $\left|A^i \cap (\varphi_n^m)^{-1}(U)\right| \geq r$, for each $1 \leq i \leq \ell$, and $U \in \varphi[B^i]$.

On the other hand, since φ is an epimorphism, for m big enough it holds that for all $A \in \mathrm{MC}(P_m)$ there is $B_A \in \mathrm{MC}(P)$ such that $\varphi_n^m[A] \subseteq \varphi[B_A]$ and, for every $U \in \varphi_n^m[A]$, one has $\left|(\varphi_n^m)^{-1}(U) \cap A\right| \geq r$.

So fix such an m, greater or equal to m_0. We construct an epimorphism $\psi : P_m \to P$ such that $\varphi\psi = \varphi_n^m$, by defining its restriction on each $A \in \mathrm{MC}(P_m)$. For $1 \le i \le \ell$, we use Lemma 4.1.3 to construct an $\mathcal{L}_R$-preserving function ψ_i from A^i onto B^i such that $\varphi\psi_i = \varphi_n^m{\restriction_{A^i}}$. Then, for each $A \in \mathrm{MC}(P_m)\setminus\{A^i \mid 1 \le i \le \ell\}$, we again use Lemma 4.1.3 to find an $\mathcal{L}_R$-preserving function ψ_A from A to B_A such that $\varphi\psi_A = \varphi_n^m{\restriction_A}$. Then, defining $\psi = \bigcup_{i=1}^{\ell} \psi_i \cup \bigcup_{A \in \mathrm{MC}(P_m)\setminus\{A^i \mid 1\le i\le \ell\}} \psi_A$, it follows that $\varphi\psi = \varphi_n^m$ and, by Lemma 3.1.8, ψ is an epimorphism. $\square$

4.2 Homogeneity properties

In this section we study some homogeneity properties of the Fraïssé fence, describing in particular its orbits under homeomorphisms. Recall that $\mathrm{Aut}(\mathbb{F}/R^{\mathbb{F}})$ is the subgroup of $\mathrm{Homeo}(\mathbb{F}/R^{\mathbb{F}})$ of homeomorphisms which preserve $\le^{\mathbb{F}/R^{\mathbb{F}}}$.

Theorem 4.2.1. *Let $J^1, \dots, J^\ell, I^1, \dots, I^\ell$ be tuples of distinct connected components of $\mathbb{F}/R^{\mathbb{F}}$. Suppose that $J^1, \dots, J^k$, $I^1, \dots, I^k$ are arc components and $J^{k+1}, \dots, J^\ell$, $I^{k+1}, \dots, I^\ell$ are singletons, for some k with $0 \le k \le \ell$. For $1 \le i \le k$, let $x^i \in J^i, y^i \in I^i$ be points which are not endpoints. Then there is $h \in \mathrm{Aut}(\mathbb{F}/R^{\mathbb{F}})$ such that $h[J^i] = I^i$, for $1 \le i \le \ell$, and $h(x^i) = y^i$ for $1 \le i \le k$.*

We obtain Theorem 4.2.1 by proving in Lemma 4.2.3 a strengthening of the converse of Proposition 1.1.2 for (F_n, γ_n^m) and using it in a back-and-forth argument which yields the desired isomorphism.

Lemma 4.2.2. *Let (P_n, φ_n^m) be a fine projective sequence in $\mathcal{F}_0$, with projective limit $\mathbb{P}$, and the quotient map $p : \mathbb{P} \to \mathbb{P}/R^{\mathbb{P}}$ be irreducible. Let $x \in \mathbb{P}/R^{\mathbb{P}}$ be such that $p^{-1}(x)$ is a singleton which is neither $\le^{\mathbb{P}}$-minimal nor $\le^{\mathbb{P}}$-maximal. For each $n \in \mathbb{N}$, let $\{x_n\} = \varphi_n[p^{-1}(x)]$. For any $n, r \in \mathbb{N}$, there is $m_0 > n$ such that for all $m > m_0$,*

$$\min\{|\{b \in P_m \mid b < x_m, \varphi_n^m(b) = x_n\}|, |\{b \in P_m \mid b > x_m, \varphi_n^m(b) = x_n\}|\} \ge r.$$

Proof. Since $p^{-1}(x)$ is neither $\le^{\mathbb{P}}$-minimal nor $\le^{\mathbb{P}}$-maximal, there is $n_0 > n$ such that x_{n_0} is neither $\le^{P_{n_0}}$-minimal nor $\le^{P_{n_0}}$-maximal. Let a, a' be the $R^{P_{n_0}}$-neighbors of x_{n_0} different from x_{n_0}. By Lemma 1.3.8 it follows that $x \in \mathrm{int}([\![x_{n_0}]\!]_{\varphi_{n_0}})$, so x has positive distance from $[\![a]\!]_{\varphi_{n_0}}$ and $[\![a']\!]_{\varphi_{n_0}}$. By Lemma 1.2.9(2), there is $m_0 > n_0$ for which the thesis holds. $\square$

Lemma 4.2.3. *Let $J^1, \dots, J^\ell$ be distinct connected components of $\mathbb{F}/R^{\mathbb{F}}$, such that $J^1, \dots, J^k$ are arcs and $J^{k+1}, \dots, J^\ell$ are singletons, where $0 \le k \le \ell$. Assume that $p^{-1}(x)$ is a singleton, for any x endpoint of some J^i. For $1 \le i \le k$, let $x^i \in J^i$ be a point which is not an endpoint, such that $p^{-1}(x^i)$ is a singleton. For each $n \in \mathbb{N}$, call $J_n^i = \gamma_n[p^{-1}(J^i)]$, and $\{x_n^i\} = \gamma_n[p^{-1}(x^i)]$. Let $P \in \mathcal{F}_0$, and $\varphi : P \to F_n$ an epimorphism. For $1 \le i \le \ell$, let $I^i \subseteq P$ be R-connected and such that $\varphi[I^i] = J_n^i$;*

assume moreover that if J^i is a singleton, then I^i is a singleton as well. For $1 \leq i \leq k$, let $y^i \in \varphi^{-1}(x^i_n)$. Then there exist $m > n$ and an epimorphism $\psi : F_m \to P$ such that:

- *$\psi[J^i_m] = I^i$ for $1 \leq i \leq \ell$;*
- *$\psi(x^i_m) = y^i$ for $1 \leq i \leq k$; and*
- *$\varphi\psi = \gamma^m_n$.*

Proof. Let $r = \max\{\left|\varphi^{-1}(a)\right| \mid a \in F_n\}$. For $1 \leq i \leq \ell$ and $m \in \mathbb{N}$, let $B^i_m \in \mathrm{MC}(F_m)$ be such that $J^i_m \subseteq B^i_m$. Let $P' \in \mathcal{F}_0$ be the structure obtained as the disjoint union of $\ell + 1$ copies of P and $\alpha : P' \to P$ be the epimorphism whose restriction to each copy of P is the identity. By (F2) there are $m' > n$ and an epimorphism $\psi' : F_{m'} \to P'$ such that $\varphi\alpha\psi' = \gamma^{m'}_n$. By Lemma 1.3.8 the endpoints of J^i belong to $\bigcup_{a \in F_{m'}} \mathrm{int}([\![a]\!]_{\gamma_{m'}})$, for $1 \leq i \leq \ell$, so we can apply Lemma 4.1.4 to find $m_0 > m'$ such that for all $m > m_0$ and $1 \leq i \leq \ell$ we obtain that $\gamma^m_{m'}[B^i_m] = J^i_{m'}$ and, if J^i is an arc, $\left|(\gamma^m_n)^{-1}(a) \cap J^i_m\right| > r$ for each $a \in J^i_n$. For $1 \leq i \leq k$, $p^{-1}(x^i)$ is a singleton and is neither $\leq^{\mathbb{F}}$-minimal nor $\leq^{\mathbb{F}}$-maximal, so by Lemma 4.2.2 there is $m_1 > m_0$ such that for all $m > m_1$ and $1 \leq i \leq k$,

$$\min\left\{\left|\left\{b \in F_m \;\middle|\; b < x^i_m, \gamma^m_n(b) = x^i_n\right\}\right|, \left|\left\{b \in F_m \;\middle|\; b > x^i_m, \gamma^m_n(b) = x^i_n\right\}\right|\right\} \geq r. \quad (4.1)$$

Now we use Lemma 4.1.3 to define, for $1 \leq i \leq \ell$, an epimorphism $\psi_i : B^i_m \to I^i$ such that $\psi_i[J^i_m] = I^i$, $\varphi\psi_i = \gamma^m_n{\restriction}_{B^i_m}$, and such that, moreover, $\psi_i(x^i_m) = y^i$ when $1 \leq i \leq k$. Let $\psi : F_m \to P$ be defined by

$$\psi(b) = \begin{cases} \alpha\psi'\gamma^m_{m'}(b) & \text{if} \quad b \notin \bigcup_{i=1}^{\ell} B^i_m \\ \psi_i(b) & \text{if} \quad b \in B^i_m \end{cases}.$$

Then $\varphi\psi = \gamma^m_n$ and ψ is an epimorphism. Indeed, ψ is $\mathcal{L}_R$-preserving by construction and for each $B \in \mathrm{MC}(P)$ there is $C \in \mathrm{MC}(F_m)$ such that $\psi'\gamma^m_{m'}[C]$ equals one of the copies of B in P', as there are more copies of B in P' than maximal chains of F_m on which ψ differs from $\alpha\psi'\gamma^m_{m'}$. $\square$

The connected components of Theorem 4.2.1 might not satisfy the hypotheses of Lemma 4.2.3, since some of the endpoints may be non-singleton $R^{\mathbb{F}}$-classes, so we cannot apply Lemma 4.2.3 directly. Therefore we first need the following lemma.

Lemma 4.2.4. *Let $\sim \subseteq R^{\mathbb{F}}$ be an equivalence relation on $\mathbb{F}$ which is the equality but on finitely many points. Then $\mathbb{F}/{\sim}$ with the induced $\mathcal{L}_R$-structure is isomorphic to $\mathbb{F}$.*

Proof. Let ℓ be the number of $\sim$-equivalence classes of cardinality greater than 1, that is, by Lemma 3.2.2, of cardinality exactly 2. Denote these equivalence classes by $\{x_1, x'_1\}, \ldots, \{x_\ell, x'_\ell\}$. To prove that $\mathbb{F}/{\sim}$ is isomorphic to $\mathbb{F}$ we show that $\mathbb{F}/{\sim}$ satisfies properties (L1), (L2) and (L3′). Inductively, it is enough to prove the assertion for $\ell = 1$. Notice also that the quotient map $q : \mathbb{F} \to \mathbb{F}/\sim$ is an epimorphism.

Property (L1) follows from (L3′) by considering, for any $P \in \mathcal{F}_0$, epimorphisms from $\mathbb{F}/\sim$ and P to a structure in $\mathcal{F}_0$ with one point.

To check that (L3′) holds, fix $P, Q \in \mathcal{F}_0$ and epimorphisms $\psi : \mathbb{F}/\sim \to P, \varphi : Q \to P$ with the objective of finding an epimorphism $\theta : \mathbb{F}/\sim \to Q$ such that $\varphi\theta = \psi$. Let $Q' \in \mathcal{F}_0$ be the structure obtained from Q by substituting each $a \in Q$ with a chain $\{a_0, a_1\}$ of length 2. In other words:

- $Q' = \{a_0, a_1 \mid a \in Q\}$;
- $R^{Q'}$ is the smallest reflexive and symmetric relation such that
  - $a_0 \, R^{Q'} \, a_1$ for every $a \in Q$,
  - $a_1 \, R^{Q'} \, a'_0$ whenever $a \, R^Q \, a'$, with $a <^Q a'$;
- $a_i \leq^{Q'} a'_j$ if and only if either $a = a', i \leq j$, or $a <^Q a'$.

Let $\chi : Q' \to Q$ be the epimorphism $a_i \mapsto a$. By (L3′) for $\mathbb{F}$ there exists $\theta' : \mathbb{F} \to Q'$ such that $\varphi\chi\theta' = \psi q$. Let $C = \theta'[\{x_1, x'_1\}]$. Let $\chi' : Q' \to Q$ be defined as

$$\chi'(a_i) = \begin{cases} a & \text{if } a_i \notin C, \\ \chi(\max C) & \text{if } a_i \in C. \end{cases}$$

Then χ' is an epimorphism using Lemma 3.1.8, which is applicable as $\forall a \in Q' \; \chi'(a_0) = a$. Define $\theta(y) = \chi'\theta'(x)$ for any $x \in q^{-1}(y)$. This is well defined as $\chi'\theta'(x_1) = \chi'\theta'(x'_1)$, and is the required epimorphism: continuity holds since for each $a \in Q$, the set $(\chi'\theta')^{-1}(a)$ is a clopen $\sim$-invariant subset of $\mathbb{F}$, so $q[(\chi'\theta')^{-1}(a)] = \theta^{-1}(a)$ is clopen in $\mathbb{F}/\sim$.

For (L2) let $\{V_1, \ldots, V_r\}$ be a clopen partition of $\mathbb{F}/\sim$. Consider the induced clopen partition $\{q^{-1}(V_1), \ldots, q^{-1}(V_r)\}$ of $\mathbb{F}$. By (L2) for $\mathbb{F}$, there exist $P' \in \mathcal{F}_0$ and an epimorphism $\varphi' : \mathbb{F} \to P'$ which refines the partition. Let $P \in \mathcal{F}_0$ be the quotient of P' which identifies a, a' if and only if $a = a'$ or $a, a \in \varphi'[\{x_1, x'_1\}]$. Then the quotient map $\psi : P' \to P$ is an epimorphism, so $\varphi(y) = \psi\varphi'(x)$ for any $x \in q^{-1}(y)$ is a well defined epimorphism. Since $\psi\varphi'$ refines $\{q^{-1}(V_1), \ldots, q^{-1}(V_r)\}$, it follows that φ refines $\{V_1, \ldots, V_r\}$. □

Proof of Theorem 4.2.1. By Lemma 4.2.4, up to considering an isomorphic structure, we can assume that the preimages of the endpoints of all the J^i's and I^i's under the quotient map $p : \mathbb{F} \to \mathbb{F}/R^{\mathbb{F}}$ are singletons, as well as the preimages of the x^i's and y^i's.

For $1 \leq i \leq \ell$, let $J^i_\infty = p^{-1}(J^i), I^i_\infty = p^{-1}(I^i)$; for $1 \leq i \leq k$, let $\{x^i_\infty\} = p^{-1}(x^i), \{y^i_\infty\} = p^{-1}(y^i)$. For each $n \in \mathbb{N}$, for $1 \leq i \leq \ell$, let $J^i_n = \gamma_n[J^i_\infty], I^i_n = \gamma_n[I^i_\infty]$;

for $1 \leq i \leq k$, let $x^i_n = \gamma_n(x^i_\infty), y^i_n = \gamma_n(y^i_\infty)$. When J^i (equivalently, I^i) is a singleton, then J^i_n, I^i_n are singletons for every $n \in \mathbb{N}$.

Let $n_0 = m_0 = 0$ and $\varphi_0 : F_{m_0} \to F_{n_0}$ be the identity. As F_0 consists of a single point, all the hypotheses of Lemma 4.2.3 are satisfied where n, P, I^i, y^i, φ of the lemma are $0, F_0, I^i_0, y^i_0, \varphi_0$, respectively. Suppose that $n_j, m_j, \varphi_j : F_{m_j} \to F_{n_j}$ have been defined and are such that $\varphi_j[I^i_{m_j}] = J^i_{n_j}$ for $1 \leq i \leq \ell$, and $\varphi_j(y^i_{m_j}) = x^i_{n_j}$ for $1 \leq i \leq k$. By Lemma 4.2.3 there exist $n_{j+1} > n_j$ and $\psi_j : F_{n_{j+1}} \to F_{m_j}$ such that $\varphi_j\psi_j = \gamma^{n_{j+1}}_{n_j}$, $\psi_j[J^i_{n_{j+1}}] = I^i_{m_j}$, for $1 \leq i \leq \ell$, and $\psi_j(x^i_{n_{j+1}}) = y^i_{m_j}$, for $1 \leq i \leq k$. Now $F_{m_j}, F_{n_{j+1}}$ and ψ_j satisfy the hypotheses of Lemma 4.2.3 with the roles of the I's and J's reversed, so there exist $m_{j+1} > m_j$ and $\varphi_{j+1} : F_{m_{j+1}} \to F_{n_{j+1}}$ such that $\psi_j\varphi_{j+1} = \gamma^{m_{j+1}}_{m_j}$, $\varphi_{j+1}[I^i_{m_{j+1}}] = J^i_{n_{j+1}}$ for $1 \leq i \leq \ell$, and $\varphi_{j+1}(y^i_{m_{j+1}}) = x^i_{n_{j+1}}$, for $1 \leq i \leq k$.

Let $\varphi, \psi : \mathbb{F} \to \mathbb{F}$ be the unique epimorphisms such that for each $j \in \mathbb{N}$, $\gamma_{n_j}\varphi = \varphi_j\gamma_{m_j}$ and $\gamma_{m_j}\psi = \psi_j\gamma_{n_{j+1}}$. Then $\varphi\psi$ and $\psi\varphi$ are the identity, so $\varphi, \psi \in \mathrm{Aut}(\mathbb{F})$. As for each $j \in \mathbb{N}$, $\gamma_{m_j}\psi[J^i_\infty] = \psi_j\gamma_{n_{j+1}}[J^i_\infty] = \psi_j[J^i_{n_{j+1}}] = I^i_{m_j}$ for $1 \leq i \leq \ell$, it follows that $\psi[J^i_\infty] = I^i_\infty$; from $\gamma_{m_j}\psi[x^i_\infty] = \psi_j\gamma_{n_{j+1}}[x^i_\infty] = \psi_j[x^i_{n_{j+1}}] = y^i_{m_j}$, it follows that $\psi[x^i_\infty] = y^i_\infty$, for $1 \leq i \leq k$. Let $h : \mathbb{F}/R^\mathbb{F} \to \mathbb{F}/R^\mathbb{F}$ be defined by $h(x) = p\psi(u)$ for any $u \in p^{-1}(x)$. Then $h \in \mathrm{Aut}(\mathbb{F}/R^\mathbb{F})$ and $h[J^i] = I^i$, for $1 \leq i \leq \ell$, and $h(x^i) = y^i$ for $1 \leq i \leq k$. $\square$

To lighten notation, let $\mathfrak{L} = \mathfrak{L}(\mathbb{F}/R^\mathbb{F}, \leq^{\mathbb{F}/R^\mathbb{F}})$, $\mathfrak{U} = \mathfrak{U}(\mathbb{F}/R^\mathbb{F}, \leq^{\mathbb{F}/R^\mathbb{F}})$.

Lemma 4.2.5. *There is $h \in \mathrm{Homeo}(\mathbb{F}/R^\mathbb{F})$ which switches $\mathfrak{U}$ and $\mathfrak{L}$.*

Proof. For any $\mathcal{L}_R$-structure A, let A^* be the $\mathcal{L}_R$-structure with the same support as A, with $R^{A^*} = R^A$ and $u \leq^{A^*} u'$ if and only if $u' \leq^A u$. Then $(A^*)^* = A$ and a function $\varphi : B \to A$ is an epimorphism from B to A if and only if it is an epimorphism from B^* to A^*. Now, if $A \in \mathcal{F}_0$, then $A^* \in \mathcal{F}_0$, so it is straightforward to check that (L1), (L2), (L3) hold for $\mathbb{F}^*$. It follows that $\mathbb{F}^*$ is the projective Fraïssé limit of $\mathcal{F}_0$ and thus that it is isomorphic to $\mathbb{F}$, via an isomorphism $\alpha : \mathbb{F} \to \mathbb{F}^*$. Let $h : \mathbb{F}/R^\mathbb{F} \to \mathbb{F}/R^\mathbb{F}$ be defined by letting $h(x) = p\alpha(u)$ for any $u \in p^{-1}(x)$. Then h is the required homeomorphism. $\square$

Corollary 4.2.6. *The Fraïssé fence is $1/3$-homogeneous. The orbits of the action of* $\mathrm{Homeo}(\mathbb{F}/R^\mathbb{F})$ *on* $\mathbb{F}/R^\mathbb{F}$ *are* $\mathfrak{L} \cap \mathfrak{U}$, $\mathfrak{L} \triangle \mathfrak{U}$, *and* $\mathbb{F}/R^\mathbb{F} \setminus (\mathfrak{L} \cup \mathfrak{U})$.

Proof. The above subspaces are clearly invariant under homeomorphisms. We conclude by Theorem 4.2.1 and Lemma 4.2.5. $\square$

The Fraïssé fence also enjoys a different kind of homogeneity property, namely that of h-homogeneity.

Proposition 4.2.7. *The Fraïssé fence is h-homogeneous.*

Proof. Fix a nonempty clopen subset U of $\mathbb{F}/R^{\mathbb{F}}$. By Lemma 3.2.7, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there is $S_n \subseteq \mathrm{MC}(P_n)$ for which $U = \bigcup_{a \in \bigcup S_n} [\![a]\!]_{\gamma_n}$. Let $Q_n = \bigcup S_n$. We prove that $(Q_n, \gamma_n^m{\restriction}_{Q_m})_{n \geq n_0}$ is a fundamental sequence in $\mathcal{F}_0$, thus showing that $p^{-1}(U)$, with the $\mathcal{L}_R$-structure inherited from $\mathbb{F}$, is isomorphic to $\mathbb{F}$, which yields the result.

Let $n \geq n_0$, $P \in \mathcal{F}_0$ and $\varphi : P \to Q_n$. Let $P' = P \sqcup (F_n \setminus Q_n)$ and $\varphi' : P' \to F_n$ be φ on P and the identity on $F_n \setminus Q_n$. Since Q_n is R^{P_n}-invariant in F_n and φ is an epimorphism, so is φ', by Lemma 3.1.8. By (F2) there are $m \geq n$ and an epimorphism $\psi' : F_m \to P'$ such that $\varphi'\psi' = \gamma_n^m$. We see that $(\gamma_n^m)^{-1}(Q_n) = Q_m$. Indeed, $\gamma_m^{-1}(Q_m) = \gamma_n^{-1}(Q_n) = p^{-1}(U)$, so $Q_m \subseteq (\gamma_n^m)^{-1}(Q_n) \subseteq \gamma_m[\gamma_n^{-1}(Q_n)] = \gamma_m[p^{-1}(U)] = Q_m$. Therefore $(\psi')^{-1}(P) = Q_m$, so $\psi = \psi'{\restriction}_{Q_m} : Q_m \to P$ is an epimorphism such that $\varphi\psi = \gamma_n^m{\restriction}_{Q_m}$. We conclude by Proposition 1.1.2. □

4.3 A strong universality property

Theorem 3.3.2 shows that any smooth fence embeds in the Cantor fence. We show a stronger universality property for the Fraïssé fence, namely that any smooth fence embeds in the Fraïssé fence via a map which preserves endpoints.

Theorem 4.3.1. *For any smooth fence Y there is an embedding $f : Y \to \mathbb{F}/R^{\mathbb{F}}$ such that $f[\mathrm{E}(Y)] \subseteq \mathrm{E}(\mathbb{F}/R^{\mathbb{F}})$. Moreover, fixing a strongly compatible order $\leq^Y$ on Y, the embedding f can be constructed so that $f[\mathfrak{L}(Y, \leq^Y)] \subseteq \mathfrak{L}$, $f[\mathfrak{U}(Y, \leq^Y)] \subseteq \mathfrak{U}$.*

Proof. By Corollary 3.4.4 there is a projective sequence (P_n, φ_n^m), with projective limit $\mathbb{P}$ such that $\mathbb{P}/R^{\mathbb{P}}$ is homeomorphic to Y, via $h : \mathbb{P}/R^{\mathbb{P}} \to Y$; moreover, h is an isomorphism between $\leq^{\mathbb{P}/R^{\mathbb{P}}}$ and $\leq^Y$. Therefore it is enough to prove the assertion for $(\mathbb{P}/R^{\mathbb{P}}, \leq^{\mathbb{P}/R^{\mathbb{P}}})$.

Let $q : \mathbb{P} \to \mathbb{P}/R^{\mathbb{P}}$ be the quotient map. We procede by induction to define a topological $\mathcal{L}_R$-structure $\mathbb{P}' \subseteq \mathbb{F}$ isomorphic to $\mathbb{P}$. Let $a_0 \in F_0$, $P_0' = \{a_0\} \subseteq F_0$, and $\theta_0 : P_0 \to P_0'$ be the unique epimorphism.

Suppose one has defined $i_n, j_n \in \mathbb{N}$, $P_n' \subseteq F_{i_n}$; assume also that, with the induced structure, $P_n' \in \mathcal{F}_0$ and there is an epimorphism $\theta_n : P_{j_n} \to P_n'$. Let $F_n' = F_{i_n} \sqcup P_{j_n}$ and $\theta_n' : F_n' \to F_{i_n}$ be the identity on F_{i_n} and θ_n on P_{j_n}. By (F2) there are $i_{n+1} > i_n$ and an epimorphism $\psi_n : F_{i_{n+1}} \to F_n'$ such that $\gamma_{i_n}^{i_{n+1}} = \theta_n'\psi_n$. Then $\psi_n^{-1}(P_{j_n})$ is an $R^{F_{i_{n+1}}}$-invariant subset of $F_{i_{n+1}}$, that is the union of a subset of $\mathrm{MC}(F_{i_{n+1}})$. Let $P_{n+1}' \subseteq \psi_n^{-1}(P_{j_n})$ be in $\mathcal{F}_0$, with respect to the induced $\mathcal{L}_R$-structure, and minimal, under inclusion, with the property that $\psi_n{\restriction}_{P_{n+1}'}$ is an epimorphism onto P_{j_n}. This means that there is a bijection $g : \mathrm{MC}(P_{j_n}) \to \mathrm{MC}(P_{n+1}')$ such that $\psi_n[g(A)] = A$ and $\left|\psi^{-1}(\min A) \cap g(A)\right| = \left|\psi^{-1}(\max A) \cap g(A)\right| = 1$, for any $A \in \mathrm{MC}(P_{j_n})$. Let $r = \max\{\left|\psi_n^{-1}(a) \cap g(A)\right| \mid a \in A, A \in \mathrm{MC}(P_{j_n})\}$.

Since the sequence (P_n, φ_n^m) is fine, by Lemma 1.2.3, there is $j_{n+1} > j_n$ such that for all $a, b \in P_{j_n}$ with $d_{R^{P_{j_n}}}(a, b) = 2$, and all $a' \in (\varphi_{j_n}^{j_{n+1}})^{-1}(a), b' \in (\varphi_{j_n}^{j_{n+1}})^{-1}(b)$, it holds that $d_{R^{P_{j_{n+1}}}}(a', b') \geq r + 1$; this means that if B is an $R^{P_{j_{n+1}}}$-connected chain in $P_{j_{n+1}}$ and $c \in \varphi_{j_n}^{j_{n+1}}[B] \setminus \{\min \varphi_{j_n}^{j_{n+1}}[B], \max \varphi_{j_n}^{j_{n+1}}[B]\}$, then $\left|(\varphi_{j_n}^{j_{n+1}})^{-1}(c) \cap B\right| \geq r$. We find an epimorphism $\theta_{n+1} : P_{j_{n+1}} \to P'_{n+1}$ by defining it on each maximal chain. Fix $B \in \mathrm{MC}(P_{j_{n+1}})$. Let $A \in \mathrm{MC}(P_{j_n})$ be such that $\varphi_{j_n}^{j_{n+1}}[B] \subseteq A$ and $B' \subseteq g(A)$ be the minimal subset such that $\psi_n[B'] = \varphi_{j_n}^{j_{n+1}}[B]$. Then $B, \varphi_{j_n}^{j_{n+1}}[B]$ and B' satisfy the hypothesis of Lemma 4.1.3(2), so there is an epimorphism $\theta_B : B \to B'$ such that $\psi_n \theta_B = \varphi_{j_n}^{j_{n+1}}\restriction_B$. Let $\theta_{n+1} = \bigcup_{B \in \mathrm{MC}(P_{j_{n+1}})} \theta_B$. Then θ_{n+1} is an epimorphism by Lemma 3.1.8: for each $A \in \mathrm{MC}(P_{j_n})$, there is $B \in \mathrm{MC}(P_{j_{n+1}})$ with $\varphi_{j_n}^{j_{n+1}}[B] = A$, so $\theta_{n+1}[B] \subseteq g(A)$, and by minimality of $g(A)$ it follows that $\theta_{n+1}[B] = g(A)$. Note that $\psi_n\restriction_{P'_{n+1}} \theta_{n+1} = \varphi_{j_n}^{j_{n+1}}$.

The functions $\gamma_{i_n}^{i_{n+1}}\restriction_{P'_{n+1}} : P'_{n+1} \to P'_n$ are epimorphisms, so $\mathbb{P}' = \{u \in \mathbb{F} \mid \forall n \in \mathbb{N}\ \gamma_{i_n}(u) \in P'_n\}$, with the induced $\mathcal{L}_R$-structure is the limit of the projective sequence $(P'_n, \gamma_{i_n}^{i_m}\restriction_{P'_m})$. Since $\gamma_{i_n}^{i_{n+1}}\restriction_{P'_{n+1}} \theta_{n+1} = \theta_n \psi_n\restriction_{P'_{n+1}} \theta_{n+1} = \theta_n \varphi_{j_n}^{j_{n+1}}$, let $\theta : \mathbb{P} \to \mathbb{P}'$ be the unique epimorphism such that for each $n \in \mathbb{N}$, $\gamma_{i_n}\restriction_{\mathbb{P}'} \theta = \theta_{n+1} \varphi_{j_{n+1}}$. Similarly, as $\varphi_{j_n}^{j_{n+1}} \psi_{n+1}\restriction_{P'_{n+2}} = \psi_n\restriction_{P'_{n+1}} \theta_{n+1} \psi_{n+1}\restriction_{P'_{n+2}} = \psi_n\restriction_{P'_{n+1}} \gamma_{i_{n+1}}^{i_{n+2}}\restriction_{P'_{n+2}}$, let $\psi : \mathbb{P}' \to \mathbb{P}$ be the unique epimorphism such that for each $n \in \mathbb{N}$, $\varphi_{j_n} \psi = \psi_n\restriction_{P'_{n+1}} \gamma_{i_{n+1}}\restriction_{\mathbb{P}'}$. Then $\theta\psi$ and $\psi\theta$ are the identity, so θ, ψ are isomorphisms. Let $f : \mathbb{P}/R^{\mathbb{P}} \to \mathbb{F}/R^{\mathbb{F}}$ be defined by letting $f(x) = p\theta(w)$ for any $w \in q^{-1}(x)$. Then f is an embedding.

We show that $\leq^{\mathbb{F}}$-maximal (respectively, $\leq^{\mathbb{F}}$-minimal) points of $\mathbb{P}'$ are $\leq^{\mathbb{F}}$-maximal (respectively, $\leq^{\mathbb{F}}$-minimal) in $\mathbb{F}$, thus concluding the proof. To this end, let $u \in \mathbb{P}'$ be $\leq^{\mathbb{F}}$-maximal in $\mathbb{P}'$ and fix $n \in \mathbb{N}$. Let $a_m = \max\{a \in P'_m \mid \gamma_{i_m}(u) \leq a\}$; by Lemma 3.5.2, there is $m > n$ such that $\gamma_{i_n}^{i_m}(a_m) = \gamma_{i_n}(u)$. By minimality of P'_m, it follows that $\psi_{m-1}(a_m)$ is $\leq^{F'_{m-1}}$-maximal, so for any $a \in F_{i_m}$ with $a_m \leq a$, we have $\psi_{m-1}(a) = \psi_{m-1}(a_m)$, so $\gamma_{i_{m-1}}^{i_m}(a) = \gamma_{i_{m-1}}^{i_m}(a_m)$. It holds therefore that $\gamma_{i_n}^{i_m}(a) = \gamma_{i_n}^{i_m}(a_m) = \gamma_{i_n}(u)$. By Lemma 3.5.2, it follows that u is $\leq^{\mathbb{F}}$-maximal in $\mathbb{F}$. The case for $\leq^{\mathbb{F}}$-minimal points is analogous. $\square$

4.4 Spaces of endpoints of the Fraïssé fence

By Lemma 4.2.5, $\mathfrak{L}$ and $\mathfrak{U}$ are homeomorphic. It also follows from that lemma that $\mathfrak{U} \setminus \mathfrak{L}, \mathfrak{L} \setminus \mathfrak{U}$ are homeomorphic. We therefore state the results in this section solely in terms of $\mathfrak{U}, \mathfrak{L} \cap \mathfrak{U}$, and $\mathfrak{U} \setminus \mathfrak{L}$, the latter of which we denote by $\mathfrak{M}$. In Theorem 4.4.7 below we see that $\mathfrak{L} \cap \mathfrak{U}$ is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$.

Corollary 4.4.1. *$\mathfrak{M}$ and $\mathfrak{L} \cap \mathfrak{U}$ are ω-homogeneous.*

Proof. From Theorem 4.2.1. $\square$

Proposition 4.4.2. *$\mathfrak{M}$ is one-dimensional.*

Proof. As $\mathfrak{M}$ is a subset of a one-dimensional space, its dimension is at most one. We now show that it is at least one. Let $x \in \mathfrak{M}$ and J be the arc component of $\mathbb{F}/R^{\mathbb{F}}$ to which it belongs. Let O be an open neighborhood of x in $\mathbb{F}/R^{\mathbb{F}}$ such that $J \not\subseteq \mathrm{cl}(O)$. Let n_0 be such that there is $B_0 \in \mathrm{MC}(F_{n_0})$ with

$$J \subseteq \bigcup_{a \in B_0} [\![a]\!]_{\gamma_{n_0}}, \quad [\![\max B_0]\!]_{\gamma_{n_0}} \subseteq O,$$

which exists by Corollary 3.5.3. Choose $a_0 \in B_0$ such that $[\![a_0]\!]_{\gamma_{n_0}} \subseteq \mathbb{F}/R^{\mathbb{F}} \setminus \mathrm{cl}(O)$ and let $a_0' \in B_0$ be the minimum such that $\bigcup_{a \geq a_0'} [\![a]\!]_{\gamma_{n_0}} \subseteq O$. Notice that $a_0 < a_0'$.

Suppose one has defined $n_i \in \mathbb{N}, B_i \in \mathrm{MC}(F_{n_i}), a_i, a_i' \in B_i$, with $a_i < a_i'$. By Lemma 4.1.1 there exists an arc component J_i of $\mathbb{F}/R^{\mathbb{F}}$ whose endpoints belong to $\mathrm{int}([\![a_i]\!]_{\gamma_{n_i}}), \mathrm{int}([\![a_i']\!]_{\gamma_{n_i}})$, respectively. By Corollary 3.5.3 there are $n_{i+1} > n_i$ and $B_{i+1} \in \mathrm{MC}(F_{n_{i+1}})$ such that

$$J_i \subseteq \bigcup_{a \in B_{i+1}} [\![a]\!]_{\gamma_{n_{i+1}}} \subseteq \bigcup_{a \in B_i} [\![a]\!]_{\gamma_{n_i}},$$
$$[\![\max B_{i+1}]\!]_{\gamma_{n_{i+1}}} \subseteq [\![a_i']\!]_{\gamma_{n_i}}.$$

Choose $a_{i+1} \in B_{i+1}$ such that $[\![a_{i+1}]\!]_{\gamma_{n_{i+1}}} \subseteq [\![a_i]\!]_{\gamma_{n_i}}$ and let $a_{i+1}' \in B_{i+1}$ be the minimum such that $\bigcup_{a \geq a_{i+1}'} [\![a]\!]_{\gamma_{n_{i+1}}} \subseteq O$, so in particular $a_{i+1} < a_{i+1}'$. Since the mesh of $([\![F_n]\!]_{\gamma_n})_{n \in \mathbb{N}}$ goes to 0, we can furthermore choose n_{i+1} so that $[\![a_{i+1}']\!]_{\gamma_{n_{i+1}}} \nsubseteq [\![a_i']\!]_{\gamma_{n_i}}$, so that in particular $a_{i+1}' \neq \max B_{i+1}$.

Let $K = \bigcap_{i \in \mathbb{N}} \bigcup_{a \in B_i} [\![a]\!]_{\gamma_{n_i}} = \lim_{i \to \infty} \bigcup_{a \in B_i} [\![a]\!]_{\gamma_{n_i}}$. By Corollary 1.2.11, K is connected, call y its maximum. We prove that

$$y \in \mathfrak{M} \quad \text{and} \quad y \in \mathrm{cl}_{\mathfrak{M}}(O \cap \mathfrak{M}) \setminus O,$$

which concludes the proof.

Since $\bigcup_{a \in B_i} [\![a]\!]_{\gamma_{n_i}} \cap [\![a_0]\!]_{\gamma_{n_0}} \neq \emptyset$ for each i, it follows that $K \cap [\![a_0]\!]_{\gamma_{n_0}} \neq \emptyset$, so $y \notin \mathfrak{L}$. Suppose there exists $y' \in \mathbb{F}/R^{\mathbb{F}}$, $y <^{\mathbb{F}/R^{\mathbb{F}}} y'$. Let U be an open set containing K while avoiding y'. There thus is $i \in \mathbb{N}$ such that $\bigcup_{a \in B_i} [\![a]\!]_{\gamma_{n_i}} \subseteq U$. For each $a' \in F_{n_i}$ with $y' \in [\![a']\!]_{\gamma_{n_i}}$, it follows that $a' \notin B_i$ as $[\![a']\!]_{\gamma_{n_i}} \nsubseteq U$. But $y \leq^{\mathbb{P}/R^{\mathbb{P}}} y'$ implies $a \leq a'$ for some $a \in B_i$, a contradiction. So $y \in \mathfrak{M}$.

Since $[\![a_i']\!]_{\gamma_{n_i}} \subseteq O$ and $\max J_i \in \mathrm{int}\left([\![a_i']\!]_{\gamma_{n_i}}\right)$ for each $i \in \mathbb{N}$, it follows that $y \in \mathrm{cl}_{\mathfrak{M}}(O \cap \mathfrak{M})$. Suppose that $y \in O$. Since y has positive distance from $K \setminus O$, there exists $i \in \mathbb{N}$ such that $y \notin \bigcup\{[\![a]\!]_{\gamma_{n_i}} \mid a \in B_i, a \leq a_i'\}$, as a_i' is the minimum element of B_i such that $\bigcup_{a \geq a_i'} [\![a]\!]_{\gamma_{n_i}} \subseteq O$, and the diameter of the $[\![a_i']\!]_{\gamma_{n_i}}$ goes to 0. It follows that $y \notin \bigcup_{a \in B_{i+1}} [\![a]\!]_{\gamma_{n_{i+1}}}$ as $\bigcup_{a \in B_{i+1}} [\![a]\!]_{\gamma_{n_{i+1}}} \subseteq \bigcup\{[\![a]\!]_{\gamma_{n_i}} \mid a \in B_i, a \leq a_i'\}$, so $y \notin K$, a contradiction. □

Corollary 4.4.3. $\mathfrak{U}$ *is* $1/2$*-homogeneous. In particular, the orbits of the action of* $\mathrm{Homeo}(\mathfrak{U})$ *on* $\mathfrak{U}$ *are* $\mathfrak{L} \cap \mathfrak{U}$ *and* $\mathfrak{M}$.

Proof. By Theorem 4.2.1, for any $x, x' \in \mathfrak{M}$, $y, y' \in \mathfrak{L} \cap \mathfrak{U}$ distinct, there is $h \in \mathrm{Aut}(\mathbb{F}/R^{\mathbb{F}})$ such that $h(x) = x', h(y) = y'$. Since $h{\restriction_{\mathfrak{U}}} \in \mathrm{Homeo}(\mathfrak{U})$, it follows that there are at most 2 orbits of the action of $\mathrm{Homeo}(\mathfrak{U})$ on $\mathfrak{U}$. Therefore it suffices to show that $\mathfrak{U}$ is not homogeneous. By Lemma 3.5.7 the space $\mathfrak{U}$ is Polish, by Proposition 3.5.5 it is not cohesive and by Proposition 4.4.2 it is not zero-dimensional. By [Dij06, Proposition 2], a Polish, non-cohesive, non-zero-dimensional space is not homogeneous. □

Proposition 4.4.4. *$\mathfrak{M}$ and $\mathfrak{L} \cap \mathfrak{U}$ are dense in $\mathbb{F}/R^{\mathbb{F}}$.*

Proof. It is easy too see that $\mathfrak{M}$ is dense in $\mathbb{F}/R^{\mathbb{F}}$ by Theorem 4.1.2.

To see that $\mathfrak{L} \cap \mathfrak{U}$ is dense, let O be a nonempty open subset of $\mathbb{F}/R^{\mathbb{F}}$ and let $n_0 \in \mathbb{N}$, $a_0 \in F_{n_0}$ be such that $[\![a_0]\!]_{\gamma_{n_0}} \subseteq O$. We define a sequence $(a_i)_{i \in \mathbb{N}}$ by induction. Suppose that n_i and $a_i \in F_{n_i}$ are defined and let $P_i = F_{n_i} \sqcup \{b\}$ and $\varphi_i : P_i \to F_{n_i}$ be the identity on F_m and $\varphi_i(b) = a_i$. By (L3′) there are $n_{i+1} > n_i$ and an epimorphism $\psi_i : F_{n_{i+1}} \to P_i$ such that $\varphi_i \psi_i = \gamma_{n_i}^{n_{i+1}}$. By Lemma 3.1.7, there exists $B_i \in \mathrm{MC}(F_{n_{i+1}})$ such that $\psi_i[B_i] = \{b\}$, so $\gamma_{n_i}^{n_{i+1}}[B_i] = \{a_i\}$. Choose $a_{i+1} \in B_i$, so $\gamma_{n_i}^{n_{i+1}}(a_{i+1}) = a_i$.

Let $u \in \mathbb{F}$ be such that $\gamma_{n_i}(u) = a_i$ for each $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, we have that $\gamma_{n_{i+1}}(u) \in B_i$ and $\gamma_{n_i}^{n_{i+1}}(\max B_i) = \gamma_{n_i}^{n_{i+1}}(\min B_i) = a_i = \gamma_{n_i}(u)$. By Lemma 3.5.2, u is both $\leq^{\mathbb{F}}$-minimal and $\leq^{\mathbb{F}}$-maximal. It follows that $p(u) \in \mathfrak{L} \cap \mathfrak{U}$. Since $\gamma_{n_0}(u) = a_0$, we have $p(u) \in [\![a_0]\!]_{\gamma_{n_0}} \subseteq O$. □

Proposition 4.4.5. *$\mathfrak{M}, \mathfrak{U}$ have the property that each nonempty open set contains a nonempty clopen subset. In particular they are not cohesive.*

Proof. The result for $\mathfrak{U}$ follows from Propositions 4.4.4 and 3.5.5.

Let O be an open subset of $\mathbb{F}/R^{\mathbb{F}}$ such that $O \cap \mathfrak{M} \neq \emptyset$. Up to taking a subset we can assume O is $\leq^{\mathbb{F}/R^{\mathbb{F}}}$-convex. By Theorem 4.1.2 there exists an arc component J of $\mathbb{F}/R^{\mathbb{F}}$ whose endpoints both belong to O, so by $\leq^{\mathbb{F}/R^{\mathbb{F}}}$-convexity, $J \subseteq O$. By Corollary 3.5.4 there exist $n \in \mathbb{N}$ and $B \in \mathrm{MC}(F_n)$ such that $J \subseteq \bigcup_{a \in B} [\![a]\!]_{\gamma_n} \subseteq O$. Since $\bigcup_{a \in B} [\![a]\!]_{\gamma_n}$ is clopen in $\mathbb{F}/R^{\mathbb{F}}$ by Lemma 3.2.6, it follows that $\bigcup_{a \in B} [\![a]\!]_{\gamma_n} \cap \mathfrak{M}$ is clopen in $\mathfrak{M}$, and it is nonempty as it contains $\max J$. □

Finally we look at $\mathrm{E}(\mathbb{F}/R^{\mathbb{F}}) = \mathfrak{L} \cup \mathfrak{U}$.

Proposition 4.4.6. *The spaces $\mathrm{E}(\mathbb{F}/R^{\mathbb{F}})$ and $\mathfrak{L} \triangle \mathfrak{U}$ are not totally separated. In fact, in $\mathfrak{L} \triangle \mathfrak{U}$ the quasi-component of each point has cardinality 2.*

Proof. Let $x \in \mathfrak{L} \triangle \mathfrak{U}$, say $x \in \mathfrak{M}$ and let z be the least element of the connected component J of x in $\mathbb{F}/R^{\mathbb{F}}$. Let U be a clopen neighborhood of x in $\mathfrak{L} \triangle \mathfrak{U}$, and let O be open in $\mathbb{F}/R^{\mathbb{F}}$ such that $U = O \cap (\mathfrak{L} \triangle \mathfrak{U})$.

If $J \nsubseteq \mathrm{cl}(O)$, from the proof of Proposition 4.4.2 it follows that there exists some $y \in \mathrm{cl}_{\mathfrak{M}}(O \cap \mathfrak{M}) \setminus O$, so

$$\emptyset \neq \mathrm{cl}_{\mathfrak{M}}(O \cap \mathfrak{M}) \setminus O \subseteq \mathrm{cl}_{\mathfrak{L}\triangle\mathfrak{U}}(O \cap \mathfrak{M}) \setminus O \subseteq \\ \subseteq \mathrm{cl}_{\mathfrak{L}\triangle\mathfrak{U}}(O \cap (\mathfrak{L} \triangle \mathfrak{U})) \setminus (O \cap (\mathfrak{L} \triangle \mathfrak{U})) = \partial_{\mathfrak{L}\triangle\mathfrak{U}}(U),$$

contradicting the fact that U is clopen in $\mathfrak{L} \triangle \mathfrak{U}$.

If $J \subseteq \mathrm{cl}(O)$ but $z \notin O$, given any open neighborhood V of z in $\mathbb{F}/R^{\mathbb{F}}$, by Theorem 4.1.2 there is some $w \in \mathfrak{M} \cap O \cap V$, so $w \in U \cap V$. This implies that $z \in \mathrm{cl}_{\mathfrak{L}\triangle\mathfrak{U}}(U) \setminus U$, contradicting again the fact that U is clopen in $\mathfrak{L} \triangle \mathfrak{U}$.

Therefore the intersection of all clopen neighborhoods of x in $\mathfrak{L} \triangle \mathfrak{U}$ also contains z. On the other hand any two points belonging to distinct components of $\mathbb{F}/R^{\mathbb{F}}$ can obviously be separated by clopen sets, so the quasi-component of x in $\mathfrak{L}\triangle\mathfrak{U}$ is $\{x, z\}$. □

Since almost zero-dimensional, T_0 spaces are totally separated, it follows that the spaces $\mathfrak{L}\triangle\mathfrak{U}$ and $\mathrm{E}(\mathbb{F}/R^{\mathbb{F}})$ are not almost zero-dimensional. This should be contrasted with Proposition 3.5.6.

We sum up what we know about the spaces of endpoints of the Fraïssé fence.

Theorem 4.4.7.

(i) *$\mathfrak{L} \cap \mathfrak{U}$ is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$.*

(ii) *$\mathrm{E}(\mathbb{F}/R^{\mathbb{F}})$ is Polish and not totally separated.*

(iii) *$\mathfrak{U}$ is $1/2$-homogeneous, Polish, almost zero-dimensional, one dimensional and not cohesive.*

(iv) *$\mathfrak{M}$ is homogeneous, strongly σ-complete, almost zero-dimensional, one dimensional and not cohesive.*

Proof.

(i) By Corollary 3.5.8 and Proposition 3.5.5, $\mathfrak{L} \cap \mathfrak{U}$ is Polish and zero-dimensional. By [Kec95, Theorem 7.7] it is enough to show that every compact subset of $\mathfrak{L}\cap\mathfrak{U}$ has empty interior. So let K be such set, and suppose toward contradiction that there is an open subset O of $\mathfrak{U}$ such that $\emptyset \neq O \cap \mathfrak{L} \cap \mathfrak{U} = O \cap \mathfrak{L} \subseteq K$. Recall that, by Proposition 4.4.4, $\mathfrak{L} \cap \mathfrak{U}$ is dense and codense in $\mathfrak{U}$. Then $O \setminus (\mathfrak{L} \cap \mathfrak{U}) = O \setminus K$ is open in $\mathfrak{U}$. Therefore, by denseness of $\mathfrak{L} \cap \mathfrak{U}$, it follows that $O \setminus (\mathfrak{L} \cap \mathfrak{U}) = \emptyset$, contradicting codenseness.

(ii) This holds by Lemma 3.5.7 and Proposition 4.4.6.

(iii) This holds by Corollary 4.4.3, Lemma 3.5.7, and Propositions 3.5.6 and 4.4.2 and 4.4.5.

(iv) This holds by Corollary 4.4.1, Remark 3.5.9, and Propositions 3.5.6 and 4.4.2 and 4.4.5.

□

A space with the properties listed in (iv) was first exhibited in [Dij06] as a counterexample to a question by Dijkstra and van Mill. We do not know however whether the two spaces are homeomorphic.

Question 4.4.8. Is $\mathfrak{M}$ homeomorphic to the space in [Dij06]?

4.5 Dynamics of the Fraïssé fence

We prove that the Fraïssé fence is approximately projectively homogeneous in the class of smooth fences and strongly compatible orders. Recall that for each $\alpha \in \operatorname{Aut}(\mathbb{F})$ there is $\alpha^* \in \operatorname{Aut}(\mathbb{F}/R^{\mathbb{F}})$ such that $p\alpha = \alpha^* p$.

Proposition 4.5.1. $\mathbb{F}$ *satisfies (SL2) with respect to* $\mathcal{F}_0$.

Proof. Let $\mathcal{U} = \{U_a \mid a \in P\}$ be a P-like open cover of $\mathbb{F}/R^{\mathbb{F}}$, for some $P \in \mathcal{F}_0$. Let $n \in \mathbb{N}$ be such that $[\![F_n]\!]_{\gamma_n}$ refines $\mathcal{U}$. Let $\varphi : F_n \to P$ be defined by mapping $a \in F_n$ to the $\leq^P$-maximal $b \in P$ such that $[\![a]\!]_{\gamma_n} \subseteq U_b$. We prove that φ is an epimorphism. By (A0), for each $b \in P$, $U_b \setminus \bigcup_{c \neq b} U_c \neq \emptyset$, so φ is surjective.

Suppose $a \mathrel{R^{F_n}} a'$, then there is $x \in [\![a]\!]_{\gamma_n} \cap [\![a']\!]_{\gamma_n}$. Then $x \in U_{\varphi(a)} \cap U_{\varphi(a')}$, so $\varphi(a) \mathrel{R^P} \varphi(a')$, by (A1).

Suppose $b \mathrel{R^P} b'$ and, without loss of generality, that $b \leq^P b'$. By condition (A1), $U_b \cap U_{b'} \neq \emptyset$. Consider $B \in \mathrm{MC}(F_n)$ such that $\bigcup_{a \in B} [\![a]\!]_{\gamma_n} \cap U_b \cap U_{b'} \neq \emptyset$. Let

$$a = \max\Big\{b \in B \;\Big|\; [\![b]\!]_{\gamma_n} \subseteq U_b, [\![b]\!]_{\gamma_n} \nsubseteq U_{b'}\Big\},$$

and a' be its immediate $\leq^{F_n}$-successor. Then $a \mathrel{R^{F_n}} a'$ and $\varphi(a) = b, \varphi(a') = b'$.

Suppose $a \leq^{F_n} a'$. By Lemma 4.1.1, there are $x \in \operatorname{int}([\![a]\!]_{\gamma_n}), x' \in \operatorname{int}([\![a']\!]_{\gamma_n})$ such that $x \leq^{\mathbb{F}/R^{\mathbb{F}}} x'$. By (A2) there are $b \leq^P b'$ such that $x \in U_b, x' \in U_{b'}$. By definition of φ, $\varphi(a)$ is either equal to, or the immediate $\leq^P$-successor of, b, and similarly for $\varphi(a')$ and b'. Since $P \in \mathcal{F}_0$, it thus holds that $\varphi(a) \leq^P \varphi(a')$.

Finally, suppose $b \leq^P b'$. By (A3) there are $x \in U_b \setminus \bigcup_{c \neq b} U_c, x' \in U_{b'} \setminus \bigcup_{c \neq b'} U_c$ with $x \leq^{\mathbb{F}/R^{\mathbb{F}}} x'$. Let $a, a' \in F_n$ be such that $x \in [\![a]\!]_{\gamma_n}, x' \in [\![a']\!]_{\gamma_n}$, so $a \leq^{F_n} a'$. For any $c \neq b$, $[\![a]\!]_{\gamma_n} \nsubseteq U_c$, since $x \in U_b \setminus \bigcup_{c \neq b} U_c$, and similarly for a', b'. Thus $\varphi(a) = b, \varphi(a') = b'$.

Then $\varphi\gamma_n : \mathbb{F} \to A$ is an epimorphism such that $[\![a]\!]_{\varphi\gamma_n} \subseteq U_a$ for each $a \in A$. □

By Theorem 3.4.3, each smooth fence $(Y, \leq^Y)$ with a strongly compatible order admits a $\mathcal{F}_0$-suitable sequence. By Theorem 1.4.6 and Corollary 1.4.7 we have the following two results.

Corollary 4.5.2. *Let $(Y, \leq^Y)$ be a smooth fence with a strongly compatible order. Let $f_0, f_1 : \mathbb{F}/R^{\mathbb{F}} \to (Y, \leq^Y)$ be epimorphims, and let $\mathcal{V}$ be an open cover of Y. Then there is $\alpha \in \mathrm{Aut}(\mathbb{F})$ such that $f_0\alpha^*$ and f_1 are $\mathcal{V}$-close, that is, for each $x \in \mathbb{F}/R^{\mathbb{F}}$ there is $V \in \mathcal{V}$ such that $f_0\alpha^*(x), f_1(x) \in V$.*

Corollary 4.5.3. $\mathrm{Aut}(\mathbb{F})$ *embeds densely in* $\mathrm{Aut}(\mathbb{F}/R^{\mathbb{F}})$.

4.5.1 $\mathrm{Aut}(\mathbb{F})$ and $\mathrm{Aut}(\mathbb{F}/R^{\mathbb{F}})$ have a dense conjugacy class

We prove that the group of automorphisms of $\mathbb{F}$ has a dense conjugacy class — a property also known as topological Rokhlin property —, then transfer the result to the automorphisms of its quotients, via Corollary 4.5.3. The existence of a dense conjugacy class in a group G implies that if $A \subseteq G$ is conjugacy invariant and has the Baire property then it is either meager or comeager.

Let S be a binary relation symobl. Let $\mathcal{F}_0^+$ be the family of $\mathcal{L}_R \cup \{S\}$ finite structures (P, S^P), such that $P \in \mathcal{F}_0$ and there are an epimorphism $\varphi : \mathbb{F} \to P$ and $\alpha \in \mathrm{Aut}(\mathbb{F})$ such that $(a, b) \in S^P$ if and only if $\alpha\big[\varphi^{-1}(a)\big] \cap \varphi^{-1}(b) \neq \emptyset$. By [Kwi14, Theorem A.1], $\mathrm{Aut}(\mathbb{F})$ has a dense conjugacy class if and only if $\mathcal{F}_0^+$ has (JPP). By [BK15, Lemma 3.11], $(P, S^P) \in \mathcal{F}_0^+$ if and only if there are $Q \in \mathcal{F}_0$ and epimorphisms $\varphi_1 : Q \to P$ and $\varphi_2 : Q \to P$ such that $S^P = \{(\varphi_1(a), \varphi_2(a)) \mid a \in Q\}$. Say that the triple $(Q, \varphi_1, \varphi_2)$ is a witness that $(P, S^P) \in \mathcal{F}_0^+$.

Given $P, P' \in \mathcal{F}_0^+$, define $P \rtimes P'$ to be the $\mathcal{L}_R \cup \{S\}$ structure with domain $P \times P'$, such that $S^{P \rtimes P'} = S^P \times S^{P'}$, $(a, a')\, R^{P \rtimes P'}\, (b, b')$ if and only if $a = b, a'\, R^{P'}\, b'$, and $(a, a') \leq^{P \rtimes P'} (b, b')$ if and only if $a = b, a' \leq^{P'} b'$.

Theorem 4.5.4. $\mathrm{Aut}(\mathbb{F})$ *has a dense conjugacy class.*

Proof. We prove that $\mathcal{F}_0^+$ satisfies (JPP). Let $P, P' \in \mathcal{F}_0^+$, with witnesses $(Q, \varphi_1, \varphi_2)$, $(Q', \varphi_1', \varphi_2')$ be given. Let $T = P \rtimes P' \sqcup P' \rtimes P$ and consider the projections θ, θ' from T to P, P', respectively.

First we prove that $T \in \mathcal{F}_0^+$. As a $\mathcal{L}_R$-structure, T is the disjoint union of a copy of P for each point of P' and a copy of P' for each point in P, so $T{\restriction_{\mathcal{L}_R}} \in \mathcal{F}_0$. For $i = 1, 2$, let $\varphi_i \times \varphi_i' : Q \rtimes Q' \to P \rtimes P'$ and $\varphi_i' \times \varphi_i : Q' \rtimes Q \to P' \rtimes P$. Then

$$Q_T = \big(Q \rtimes Q' \sqcup Q' \rtimes Q,\ \varphi_1 \times \varphi_1' \sqcup \varphi_1' \times \varphi_1,\ \varphi_2 \times \varphi_2' \sqcup \varphi_2' \times \varphi_2\big)$$

is a witness for $T \in \mathcal{F}_0^+$. Indeed, $Q_{T{\restriction}\mathcal{L}_R} \in \mathcal{F}_0$ and the two maps are epimorphisms since $(a, a')\, R^{P \rtimes P'}\, (b, b')$ if and only if $a = b$ and $a'\, R^{P'}\, b'$ if and only if there are $c \in Q, c', d' \in Q'$ with $c'\, R^{Q'}\, d'$ such that $\varphi_1(c) = a = b, \varphi_1'(c') = a', \varphi_1'(d') = b'$ if and only if $(c, c')\, R^T\, (c, d')$ and $\varphi_1 \times \varphi_1'(c, c') = (a, a'), \varphi_1 \times \varphi_1'(c, d') = (a, b') = (b, b')$. Also, $(a, a')\, S^{P \rtimes P'}\, (b, b')$ if and only if $a\, S^P\, b$, $a'\, S^{P'}\, b'$ if and only if there exist $c \in Q, c' \in Q'$ such that $\varphi_1(c) = a, \varphi_2(c) = b, \varphi_1'(c') = a', \varphi_2'(c') = b'$ if and only if $\varphi_1 \times \varphi_1'(c, c') = (a, a'), \varphi_2 \times \varphi_2'(c, c') = (b, b')$. The other case is symmetric.

Notice that θ, θ' are $\mathcal{L}_R$-epimorphisms by Lemma 3.1.8, since maximal chains are either mapped to a point or onto an isomorphic maximal chain. Moreover $\theta^{(2)}(S^T) = \theta^{(2)}(S^P \times S^{P'} \sqcup S^P \times S^{P'}) = \theta \times \theta(S^P \times S^{P'}) = S^P$, and similarly $\theta'^{(2)}(S^T) = S^{P'}$. $\square$

By Corollary 4.5.3 and Theorem 4.5.4, we have the following.

Corollary 4.5.5. $\mathrm{Aut}(\mathbb{F}/R^{\mathbb{F}})$ *has a dense conjugacy class.*

We wonder whether the results from [BK19] can be adapted to unveil more dynamical information on the Fraïssé fence and its prespace, particularly regarding the universal minimal flows of their groups of automorphisms. It would therefore be interesting to investigate whether the Ramsey results on the family of finite structures approximating the Lelek fan can be generalized to $\mathcal{F}_0$.

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